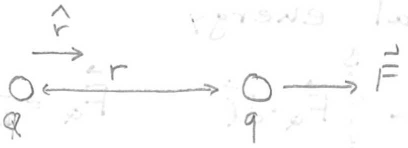
Electricity 2022

- Coulombs law $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}$



ε_0 - permitivity of free space

- Principle of superposition
  - ↳ force on q due to charges $Q_1, Q_2, Q_3 \dots$ can be written as

$$\vec{F} = \frac{q}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{Q_i}{r_i^2}\hat{r}_i$$

↳ this implies that charges excert forces on eachother and is incor

- Electric fields
  - ↳ the charges create a field which then excerts a force on the charge q

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r} \qquad \vec{F} = q\vec{E} \qquad \text{superposition: } \vec{E} = \sum_i^n \vec{E}_i$$

- Field lines

↳ a line tangent to $\vec{E}$ at every point

↳ field $\vec{E}$ and line element $d\vec{l}$ must be parallel

so $dx = E_x \quad dy = E_y \quad dz = E_z$ solved as: $\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z}$

↳ don't actually exist only used to represent the field and can not be superimposed

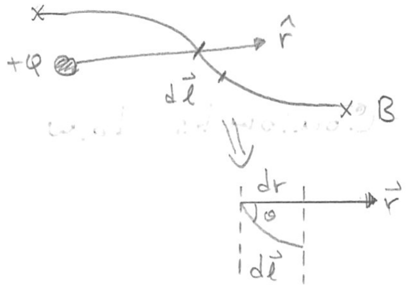
- Potential energy
  - ↳ apply a force $\vec{F}_{ext}$ to move q from A to B at a constant speed in the field due to Q

work done by $\vec{F}_{ext}$ $\quad W_{ext} = \int_A^B \vec{F}_{ext}\cdot d\vec{l} = -\int_A^B \vec{F}_Q \cdot d\vec{l}$

$$W_{ext} = \Delta U$$

↳ since $\vec{E}$ field is conservative ΔU is path independant

Potential energy



$$W_{ext} = -\int_A^B \vec{F}_Q \cdot d\vec{l} \qquad \vec{F}_Q = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r} \qquad \hat{r}\cdot d\vec{l} = dr$$

$$W_{ext} = \Delta U_{AB} = \frac{qQ}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

Potential difference

$$\Delta V_{AB} = -\int_A^B \frac{\vec{F}_Q}{q}\cdot d\vec{l} = -\int_A^B \vec{E}\cdot d\vec{l} \qquad \Delta V_{AB} = \frac{\Delta U_{AB}}{q} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

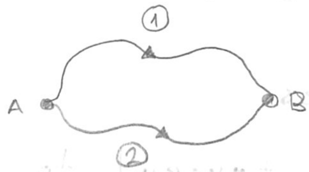
↳ also path idependant

↳ $V=0$ can be chosen arbitrarily (adding constant = gauge transformation)

- Potential at a point P

$$V = -\int_\infty^P \vec{E}\cdot d\vec{l} \qquad V(r) = \frac{Q}{4\pi\varepsilon_0 r}$$

↳ gives a scalar field



- Electrostatic circulation law

$$\Delta V_{AB} = -\int_{A\,(1)}^{B} \vec{E}\cdot d\vec{l} = \int_{B\,(2)}^{A} \vec{E}\cdot d\vec{l} \qquad \oint \vec{E}\cdot d\vec{l} = 0 \;\Longrightarrow\; \nabla\times\vec{E} = 0$$

using Stokes theorem

↳ for any closed loop only true in electrostatics !

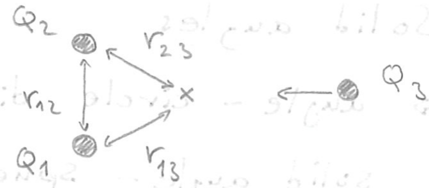
- Binding energy of 2 charges $U = Q_2 V(r_{12}) = \dfrac{Q_1 Q_2}{4\pi\varepsilon_0 r_{12}}$

↳ U is the property of the whole system and is contained within the electric field

- electron volts $1\,eV = 1.602\times10^{-19}\,J$

↳ change in energy when charge of one e^- is accelerated through a potential difference of 1V

- potential energy of a set of charges



↳ move Q_3 from ∞ to point P with a force $\vec{F}_{ext}$ at a constant speed

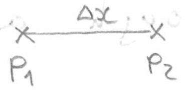
↳ overall we get $$U = \sum_{i=1}^{n} \frac{1}{2} Q_i V_i$$

- equipotential surfaces

$\Delta V_{AB} = -\int_A^B \vec{E} \cdot d\vec{l}$ so if $d\vec{l}$ is $\perp$ to $\vec{E}$ $\Delta V_{AB} = 0$

↳ equipotential surface is perpendicular to field $\vec{E}$

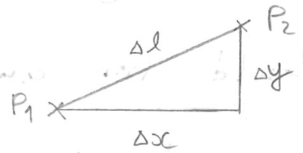
- 1D potential



$P_1 = x$
$P_2 = x + \Delta x$

$$V(P_2) = V(x+\Delta x) = V(P_1) + \Delta x \frac{dV}{dx} + \dots$$

- 2D potential



$\Delta \vec{l} = \Delta x \, \hat{x} + \Delta y \, \hat{y}$

$$V(P_2) = V(P_1) + \Delta x \frac{\partial V}{\partial x} + \Delta y \frac{\partial V}{\partial y}$$

- 3D potential

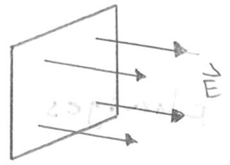
$\Delta \vec{l} = \Delta x \, \hat{x} + \Delta y \, \hat{y} + \Delta z \, \hat{z}$ $\quad V(P_2) = V(P_1) + \Delta x \frac{\partial V}{\partial x} + \Delta y \frac{\partial V}{\partial y} + \Delta z \frac{\partial V}{\partial z}$

$\Delta V = -\vec{E} \cdot \Delta \vec{l} = -E_x \Delta x - E_y \Delta y - E_z \Delta z$ and $E_x = -\frac{\partial V}{\partial x}$ $\quad E_y = -\frac{\partial V}{\partial y} \dots$

so $\vec{E} = -\nabla V$

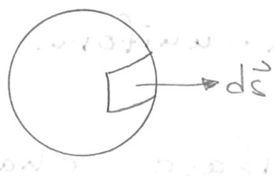
↳ now identity $\nabla \times (\nabla A) = 0$ gives electrostatic circulation law

- Electric flux $\quad \phi_{\vec{E}} = \iint_S \vec{E} \cdot d\vec{s}$

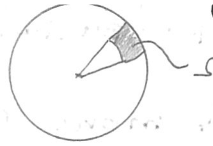


↳ can be interpreted as no of field lines passing through the surface

↳ by convention for closed surface $d\vec{s}$ points out



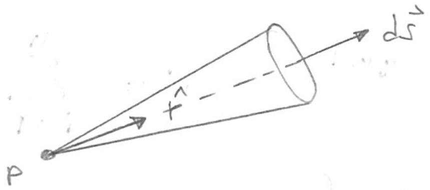
- Solid angles



↳ angle - circle divided into 2π radians

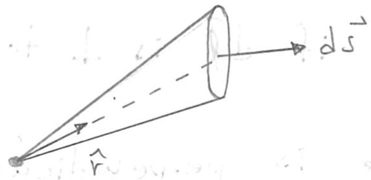
solid angle - sphere divided into 4π steradians

↳ for a spherical surface where $d\vec{s}$ is // to $\hat{r}$



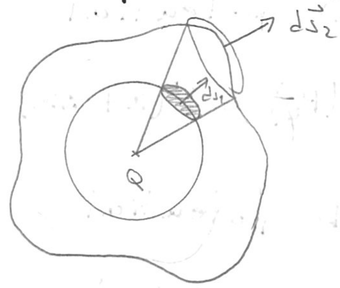
$$d\Omega = \frac{1}{r^2} ds$$

↳ for a general surface where $d\vec{s}$ and $\hat{r}$ are not alligned



$$d\Omega = \frac{d\vec{s} \cdot \hat{r}}{r^2}$$

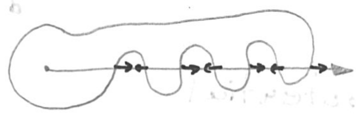
· electric flux through a closed surface due to point charge



$$\phi_E = 4\pi r^2 \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2} = \frac{Q}{\varepsilon_0}$$

↳ can be shown to hold for any surface

↳ folds in surface are ok since normals to surface are in opposite direction so cancel out



- Gauss's law

↳ use superposition to find ϕ_E through arbitrary closed surface containing charges $Q_1, Q_2, Q_3 \ldots$

$$\oiint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0} \xrightarrow[\text{divergence theorem}]{\text{charge density}} \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

use to find E

↳ all charges contribute to electric field $\vec{E}$ but only charges inside a surface contribute to flux

- non-uniform charge density $\quad Q = \iiint_V \rho \, dV$

- surface charge density $\quad Q = \iint_S \sigma \, ds$

- inside a uniformly charged sphere

↳ charge Q and radius a

$\rho = \frac{Q}{\frac{4}{3}\pi a^3}$ q enclosed: $q(r) = \frac{Qr^3}{a^3}$ $E(r) = \frac{Qr}{4\pi\varepsilon_0 a^3}$ for $r < a$

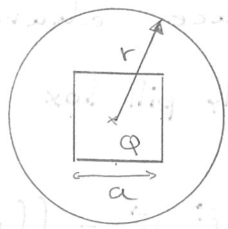
↳ so inside $E(r) \propto r$

- outside a uniformly charged sphere

$q(r) = Q$ $E(r) = \frac{Q}{4\pi\varepsilon_0 r^2}$ for $r > a$

↳ so outside $E(r) \propto \frac{1}{r^2}$

- non-uniform surface



↳ surface integral $= \frac{Q}{\varepsilon_0}$

↳ $\vec{E}$ is not uniform so stronger near corners

- Poisson's equation $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

↳ alternative way of writing Gauss law

rewrite $\vec{E}$ in terms of potential V $\vec{E} = -\nabla V$

$\nabla \cdot \vec{E} = -\nabla \cdot (\nabla V) = -\nabla^2 V$

↳ if there is no charge density $\rho = 0$ becomes Laplace equ $\nabla^2 V = 0$

- Uniqness theorem

↳ electrostatics can be written in two forms

$\vec{E} = -\nabla V$ $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

or

$\nabla \times \vec{E} = 0$ $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$

- conductor

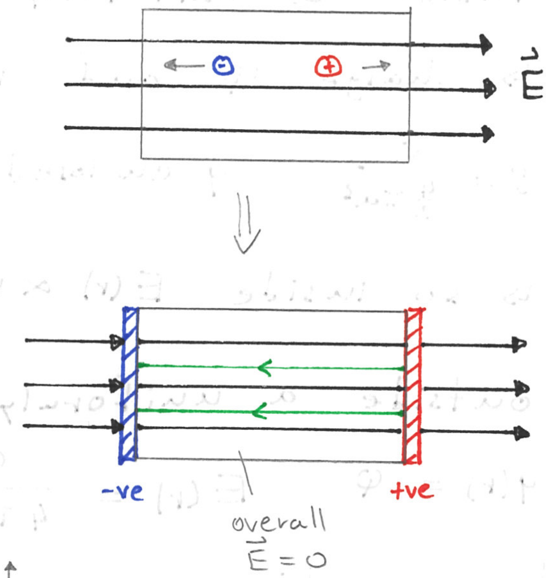
↳ has free moving charged particles

↳ electric field inside is allways 0

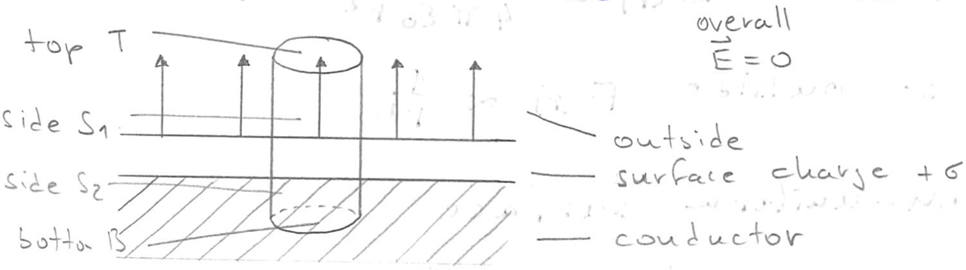
- insulator

↳ no free moving charged particles

- electric field inside conductor $E = 0$
  - ↳ if an electric field $\vec{E}$ is applied e^- move in response
  - ↳ this creates a surface charge due to e^- moving to one side
  - ↳ the $\vec{E}$ due to surface charge is equal and opposite to applied $\vec{E}$
  - ↳ overall $\vec{E} = 0$



- Surface charge
  - ↳ apply pill box analysis



$$\oiint \vec{E} \cdot d\vec{s} = \iint_T \vec{E} \cdot d\vec{s} + \iint_{S_1} \vec{E} \cdot d\vec{s} + \iint_{S_2} \vec{E} \cdot d\vec{s} + \iint_B \vec{E} \cdot d\vec{s} = EA$$

T: non zero since $\phi_E = EA$; B: zero since $\vec{E} = 0$ in conductor

; S_1: zero since $\vec{E} \cdot \vec{n} = 0$; S_2: zero since $\vec{E} = 0$ in conductor

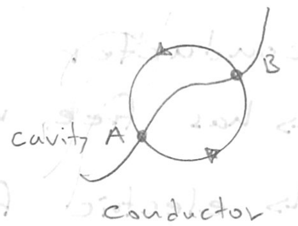
$$E = \frac{\sigma}{\varepsilon_0} \quad \sigma - \text{surface charge density}$$

↳ for curved surface make pill box infinitesimal $A \to dA$

- electrostatic shielding (Faraday cage)
  - ↳ inside empty cavity of a conductor $\vec{E} = 0$

for any electric field $\oint \vec{E} \cdot d\vec{\ell} = 0$

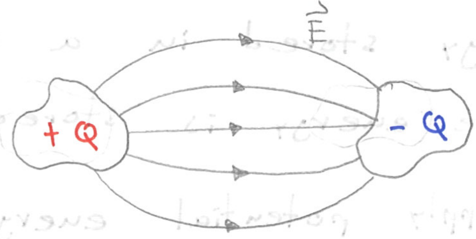
$$\oint \vec{E} \cdot d\vec{\ell} = \underbrace{\int_A^B \vec{E} \cdot d\vec{\ell}}_{\text{conductor}} + \underbrace{\int_B^A \vec{E} \cdot d\vec{\ell}}_{\text{cavity}}$$



$$0 = 0 + \int_B^A \vec{E} \cdot d\vec{\ell} \quad \text{so } \vec{E} = 0$$

1st integral is 0 since $\vec{E}$ is inside conductor

- capacitor basics



↳ consider an isolated pair of conductors A, B charged to $\pm Q$

↳ isolated so all field lines from A end at B

↳ all charge is confined to the surface

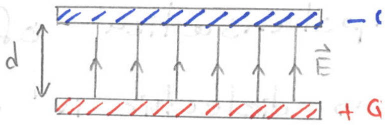
↳ conductor A has potential V_A and B V_B $\quad V = V_A - V_B$

↳ if Q is doubled $\sigma \to 2\sigma$ and $V_{A,B} \to 2V_{A,B}$ so $V \to 2V$

Q is proportional to V

↳ define capacitance C $\quad Q = CV \quad C = \frac{Q}{V}$

- parallel plate capacitor



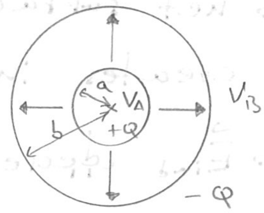
↳ 2 large plates area A separated by d

ignore edge effects charge $\pm Q$ uniformly spread on surface

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A} \qquad V = \frac{Qd}{\varepsilon_0 A} \qquad C = \frac{Q}{V} \quad \text{so} \quad C = \frac{\varepsilon_0 A}{d}$$

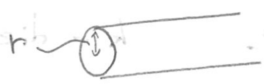
- spherical capacitor

↳ apply gauss's law to get $V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$



$$C = \frac{4\pi\varepsilon_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

- capacitance of a single conductor



↳ let $b \to \infty$ $\quad C = \frac{4\pi\varepsilon_0}{\frac{1}{a} - 0} \qquad C = 4\pi\varepsilon_0 r$

↳ energy is stored in the field surrounding the conductor

- energy stored in a capacitor

↳ the energy is stored in form of $\vec{E}$ field between plates

↳ apply potential energy of set of charges $U = \sum_{i=1}^{n} \frac{1}{2} Q_i V_i$

$$U = \frac{1}{2} Q_A V_A - \frac{1}{2} Q_B V_B = \frac{1}{2} Q (V_A - V_B)$$

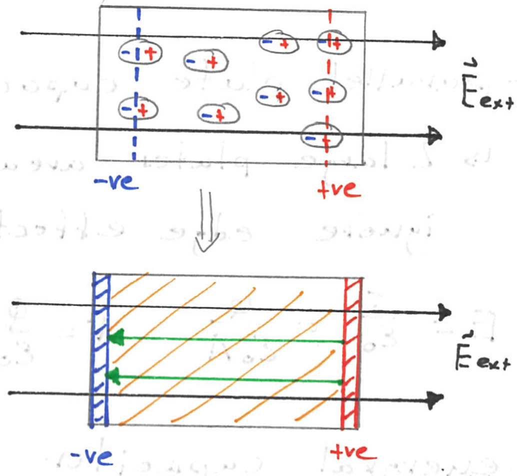
$$U = \frac{1}{2} Q V = \frac{1}{2} C V^2$$

↳ $E = \frac{Q}{\varepsilon_0 A} \Rightarrow Q = \varepsilon_0 A E \quad V = Ed$

so $U = \frac{1}{2}(\varepsilon_0 A E)(Ed) = \frac{1}{2} \varepsilon_0 E^2 (Ad)$

↳ define energy density $\underline{u = \frac{1}{2} \varepsilon_0 E^2}$ - for any field !

- polarisation of dielectric



↳ dielectric is an insulator with polar molecules free to move around

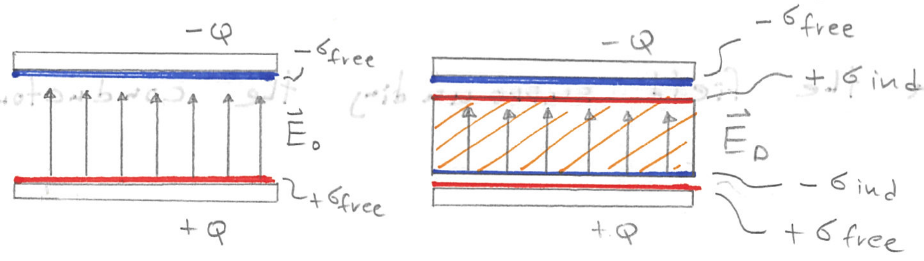
↳ apply external electric field $\vec{E}_{ext}$ which causes dipoles to allign

↳ net surface charge established creates induced field in dielectric $\vec{E}_{ind}$ all occours within dielectr

↳ $\vec{E}_{ind}$ opposes $\vec{E}_{ext}$ so overall field is weaker than it would be in a vacuum

$$\vec{E}_D = \frac{\vec{E}_0}{k}$$ k - dielectric constant usually $k > 1$

- dielectrics in capacitors



$E_0 = \frac{\sigma_{free}}{\varepsilon_0} \qquad E_D = \frac{\sigma_{free}}{k\varepsilon_0} = \frac{\sigma_{free}}{\varepsilon} \qquad \varepsilon = k \varepsilon_0$

↳ apply pill box analysis and Gauss's law

$E_D A = \frac{A(\sigma_{free} - \sigma_{ind})}{\varepsilon_0}$ so $E_D = \frac{\sigma_{free} - \sigma_{ind}}{\varepsilon_0}$ and $E_D = \frac{\sigma_{free}}{k\varepsilon_0}$

$\sigma_{ind} = \sigma_{free}\left(1 - \frac{1}{k}\right)$

↳ procedure for analysis: charge + isolate capacitor then introduce dielectri

- effects of dielectric on capacitor properties

↳ capacitance $C_D = k C_0$ so increases $\propto k$

↳ same charge can be displaced with a smaller pd

↳ energy stored $U_D = k U_0$ so increases $\propto k$