Determining the Gravitational Acceleration Constant Using a Stiff Pendulum

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***Abstract*—Equation describing the motion of a simple pendulum is derived and it is shown that it can be used to determine gravitational acceleration constant g. Equation for a torsional pendulum is derived and used to find the moment of inertia of the the stiff pendulum by comparison to a uniform cylindrical rod whose moment of inertia is determined. The experimental method is described and discussed. The data obtained is analyzed using Python to obtain a value and uncertainty of g. The sources, propagation and effects of the uncertainties are then discussed.**

I. Introduction

THE gravitational acceleration constant g is the acceleration of a body experiencing free fall. A usual use is in the context of small light bodies in close proximity to the surface of large massive bodies such as planets.

There exist numerous methods of determining g. The one discussed here is significant due to the rudimentary theoretical background as well as a simple procedure. It involves only basic equipment, making it very accessible, while still providing accurate results.

II. Theory

A. Period of a Simple Stiff Pendulum

A torque of a force can be defined by (1) [1]. Based on the geometry of the setup in Fig. 1, equation (1) can be modified by using the component of weight perpendicular to the pendulum to give (2), where mg is the weight and h is the distance from the pivot to the centre of mass. Newtons second law for rotation (3) states that the sum of torques about principal axis of is equal to the product of rotational inertia and angular acceleration [2].

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{1}$$

$$\tau = -mgh\sin\theta \tag{2}$$

$$\sum \vec{\tau} = I\vec{\alpha} \tag{3}$$

By combining (2) and (3) a final differential equation (DE) for the simple stiff pendulum is derived as (4). Equation (4) can be rewritten, using the small angle approximation $\sin\theta \approx \theta$, as (5) which represents simple harmonic motion (SHM).

$$\alpha = \frac{mgh}{I_P}\sin\theta \tag{4}$$

$$\alpha = \frac{d^2\theta}{dt^2} \approx \frac{mgh}{I_S}\theta \tag{5}$$

The solution to (5) involves angular frequency ω which is related to the properties of the stiff pendulum by (6). Equation

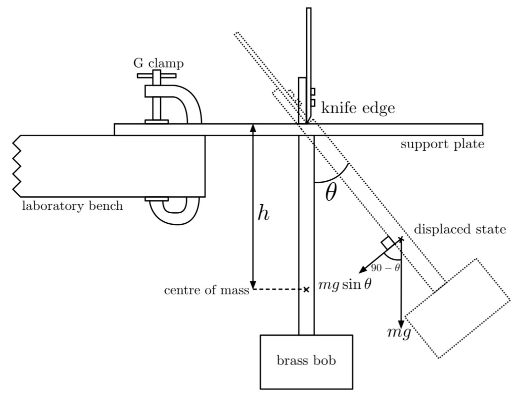

Fig. 1. Diagram showing a side view of the simple stiff pendulum experimental setup and its geometry.

(6) can be rewritten in terms of period of oscillations T_S to give (7) [3]. Finally by rearranging (7) to make g the subject we obtain the final equation (8) [4].

$$\omega = \sqrt{\frac{mgh}{I}} \tag{6}$$

$$T_S = 2\pi\sqrt{\frac{I_S}{mgh}} \tag{7}$$

$$g = \left(\frac{2\pi}{T_S}\right)^2 \frac{I_S}{mh} \tag{8}$$

B. Moments of Inertia

A torsional pendulum is used to determine the moment of inertia of the stiff pendulum I_P in (6). A suspending wire acts as a torsional spring assumed to obey the angular Hooke's law (9), which is only valid for a small angular displacement. Equation (3) is combined with (9) to arrive at (10), another DE representing SHM. By same logic as previously the solution to (10) leads to (11) [4] which relates the period of oscillation T_P to the moment of inertia and the torsional constant of the wire.

$$\tau \approx -\kappa\theta \tag{9}$$

$$\alpha = \frac{d^2\theta}{dt^2} \approx -\frac{\kappa\theta}{I} \tag{10}$$

$$T_P = 2\pi\sqrt{\frac{I_P}{\kappa}} \tag{11}$$

The torsional constant of the wire used is unknown. Therefore the torsional pendulum experiment is repeated with a uniform cylindrical rod. The period of oscillation is T_R and a moment of inertia I_R. The ratio of the periods is equal to the square of the ratio of moments of inertia as shown in (12) and is rearranged to (13) [4].

$$\frac{T_P}{T_R} = \sqrt{\frac{I_P}{I_R}} \tag{12}$$

$$I_P = I_R \left(\frac{T_P}{T_R}\right)^2 \tag{13}$$

I_R is found using the definition of a moment of inertia (14) [5]. ρ is the linear mass density and since the rod is assumed to be uniform it is constant as shown in (15). The centre of mass is now set to be at $x = 0$ so the integral over the entire length of the rod (14) takes the form of (16).

$$I = \int \rho r^2 dr \tag{14}$$

$$\rho = \frac{M_R}{L} \tag{15}$$

$$\begin{aligned} I_R &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{M}{L} r^2 dr \\ &= \frac{1}{12} M_R L^2 \end{aligned} \tag{16}$$

Equation (16) can now be substituted into (13) to express the moment of inertia of the stiff pendulum about its centre of mass in terms of measured quantities as (17). Applying the parallel axis theorem to (17) gives (18) which relates the moment of inertia about the pivot of the simple pendulum to the moment of inertia about the centre of mass of the stiff pendulum.

Equation (17) is now obtained by substituting (16) into (13). The parallel axis theorem [5] gives (18) which relates I_P, where the axis of rotation is through the centre of mass, to I_S, where the axis of rotation are through the knife edge which is the pivot of the simple pendulum.

$$I_P = \frac{1}{12} M_R L^2 \left(\frac{T_P}{T_R}\right)^2 \tag{17}$$

$$I_S = I_P + M_P h^2 \tag{18}$$

C. Deriving the Final Expression

The final equation (20), which expresses g in terms of only measured quantities, is now found by simply substituting (17) into (18).

$$g = \frac{h}{12} \left(\frac{2\pi}{T_S}\right)^2 \left(\frac{M_R}{M_P} \left(\frac{T_P L}{T_R h}\right)^2 + 12\right) \tag{19}$$

III. Experimental Method

The experimental method described bellow closely follows the Year 1 Lab Manual by S. P. D. Mangles [4].

A. Simple Stiff Pendulum

The simple pendulum was set up by attaching an overhanging aluminium support plate to a laboratory bench using a G-Clamp. A stiff pendulum composed of an aluminium knife edge and a cylindrical brass bob connected to either side of a cylindrical rod was suspended, through a cutout, from the support plate by the knife edge as shown in Fig. 1.

The pendulum was displaced by a small angle θ and released, carefully not to exert a force on the pendulum. Using a timer the time t_S to complete 20 oscillations was measured and recorded and the measurement was repeated 20 times.

B. Torsional Pendulum

The torsional pendulum was set up, first using a uniform cylindrical aluminium rod suspended at its centre of mass by a solid copper wire of diameter $0.7\ mm$ [4], tied to a support bar clamped to the laboratory desk. This setup is identical to the one in Fig. 2, except the object suspended is the cylindrical rod. The setup was left to reach equilibrium during which the the copper wire is under no torsional stress.

The rod was then displaced by a small angle θ about the copper wire. The rod was carefully released not to exert a force on the rod and a timer was started. The time t_R required for the rod to complete 5 oscillations was measured and recorded. The measurement was then repeated 10 times.

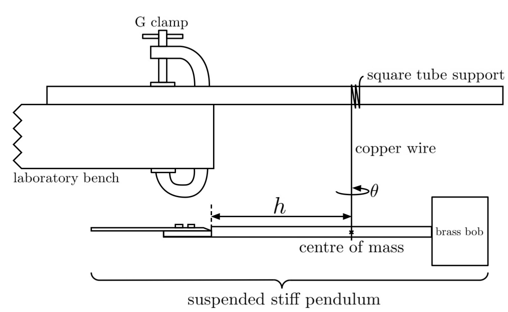

Fig. 2. Diagram showing a side view of the experimental setup for the torsional pendulum with suspended stiff pendulum.

The position of the centre of mass of the stiff pendulum is determined by suspending it on the top edge of a triangular aluminium prism and adjusting until the pendulum remains balanced (suspended horizontally for at least a few seconds). The point of contact with the prism was then marked by a line as the centre of mass.

The cylindrical rod was removed from the torsional pendulum and replaced with the stiff pendulum suspended from its centre of mass as shown in Fig. 2 and the setup was left to equilibrate. During the replacement, care was taken to prevent pinching or adjusting the length of the wire as this would change its torsional constant.

The experiment was then repeated in exactly the same way as for the cylindrical rod. The time t_S to complete 5 oscillations was recorded and this measurement was repeated 10 times.

C. Other Measurements

A meter rule was used to measure the length L of the uniform cylindrical rod and the distance h from the knife edge to the marked center of mass on the stiff pendulum.

The mass M_R of the uniform cylindrical rod and the mass M_P of the stiff pendulum were measured using a digital top pan balance and recorded.

IV. Results

The raw data as well as images of the setup and the Python code used for data analysis can be found at the following GitHub repository.

A. Numerical Values of Quantities from Collected Data

The values of the mean period of oscillation T_S, T_R, T_P are calculated from the N measurements of the time to complete n oscillations t_S, t_R, t_P. This is done by finding the arithmetic mean of all of the repeated measurements and then dividing it by both the number of measurements N and number of oscillations timed n as shown in (20).

$$T = \frac{1}{nN} \sum^{N} t \tag{20}$$

The uncertainty in the calculated periods of oscillation is determined by finding the standard error on the mean SE [6]. First the sample standard deviation of measured times t is found and n is subtracted to account for the fact that values of t must be divided by n. Finally it is divided by $\sqrt{N}$.

$$SE = \frac{s_t - n}{\sqrt{N}} \tag{21}$$

The absolute uncertainties in the directly measured quantities, T_S, T_R, T_P, are the uncertainties due to the accuracy of the measuring instrument and method used.

TABLE I: Numerical Values

Symbol	Value	SI Unit
T_S	1.7120 ± 0.0009	s
T_R	10.32 ± 0.02	s
T_P	24.19 ± 0.03	s
M_R	0.26832(1)	kg
M_P	0.74678(1)	kg
L	0.395 ± 0.001	m
h	0.532 ± 0.002	m

Table containing all numerical values and their uncertainties.

B. Calculation of the Value and Uncertainty of g

By substituting values from Tab. I into (19) the value of gravitational acceleration constant g is determined to be $7.82 \pm 0.03\ m \cdot s^{-2}$.

The uncertainty δg is found using the rule for propagation of uncertainties in a function of several variables (22) [6] where the sum ranges over all of the variables of the function of g represented by q_i. This approach assumes the errors in measurements are random and is justified in the later sections. The packages SymPy and NumPy are used to find and evaluate the partial derivatives.

$$\delta g = \sqrt{\sum_i \left(\frac{\partial g}{\partial q_i} \delta q_i \right)^2} \tag{22}$$

The accepted value for the gravitational acceleration constant in the city of London where the experiment was preformed is $9.816\ m \cdot s^{-2}$ [7] showing a percentage deviation of 20.3% from the obtained value of $7.82 \pm 0.03\ m \cdot s^{-2}$.

C. Sources of Uncertainties and Improvements

In the simple pendulum setup, the knife edge is free to slide around on top of the support plate, giving rise to a significant random error. To minimize this error the pendulum must only be displaced by small angles and preferably the pendulum should be suspended using light ceramic bearings instead of the knife edge. As T_S is small the effects of human judgment/reflex in determining the end point of oscillation contribute significantly to the uncertainty. Both of the discussed sources of uncertainty give rise to random errors.

For both of the setups involving the torsional pendulum it was difficult to release the pendulum without initiating unwanted oscillations. This could be improved by using a magnetic release mechanism. Due to the low restoring torque provided by the wire the system was difficult to stabilize and prone to disruptions from the surroundings such as air currents leading to a significant random error. An attempt was made to decrease these effects by decreasing the length of the wire to increase its torsional coefficient. Ideally a torsion spring of a greater torsional coefficient and elastic limit should be used.

In (19) the masses of the rod and the stiff pendulum appear as a fraction therefore any errors directly proportional to the mass will cancel out. The balance was checked for zero error and has a relatively high precision therefore the uncertainty in masses does not significantly contribute to the overall uncertainty.

The distance h has a greater absolute uncertainty than L since the measurement also involved determining the cetre of mass of the stiff pendulum. The region of the centre of mass was marked with a line approximately $1\ mm$ wide, so uncertainty in h was increased to accommodate for this. The contribution of uncertainties of L and h to the overall uncertainty are significant but lower than those of the periods of oscillation.

V. Conclusion

Mathematical models for the motion of a simple and a torsional pendulum has been developed. The solutions of these models have been combined to give an equation for the gravitational acceleration constant g in terms of measurable quantities. Experiments were then carried out to determine the values of the required quantities. The values were used to calculate g and its uncertainty to a reasonable degree of experimental accuracy. The experiment should now be repeated with the suggested improvements and compared to this one to see the effects of the improvements.

References

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