Functions 2021

- Heaviside function

$$H(x) = \begin{cases} 0 & (x<0) \\ 1 & (x \geq 0) \end{cases}$$

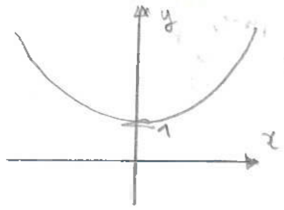
- even function $f(x) = f(-x)$ e.g.) x^2, $\cos(x)$
- odd functions $f(x) = -f(-x)$ e.g.) x^3, $\sin(x)$, $\tan(x)$
  ↳ composition of two odd functions also odd e.g.) $\sin(x^3)$
- not all functions odd/even but all can be written as their combinat

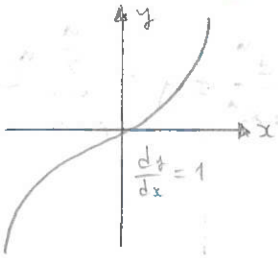
$$g(x) \equiv \underbrace{\tfrac{1}{2}\left(g(x)+g(-x)\right)}_{\text{even}} + \underbrace{\tfrac{1}{2}\left(g(x)-g(-x)\right)}_{\text{odd}}$$

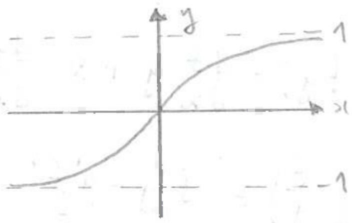
- composite functions dont commute $f(g(x)) \neq g(f(x))$

- hyperbolic functions

$\cosh x = \dfrac{e^x + e^{-x}}{2}$ $\quad \sinh x = \dfrac{e^x - e^{-x}}{2}$ $\quad \tanh = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$







- Osbornes rule $\cosh(ix) \equiv \cos(x)$ $\quad \sinh(ix) \equiv i\sin(x)$
  ↳ whenever theres a product of $\sin(x)$ functions the hyperbolic identity has an opposite sign

- inverse hyperbolics

$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$ $\quad \cosh^{-1}(x) = \pm \ln\left(x + \sqrt{x^2-1}\right)$ $\quad \tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

- arithmetic mean (AM) $\frac{\sum_i a_i}{n}$

- geometric mean (GM) $\sqrt[n]{\prod_i a_i}$

- Cauchy - Schwartz inequality $\left(\sum_i^n a_i^2\right)\left(\sum_i^n b_i^2\right) \geq \left(\sum_i^n a_i b_i\right)^2$
  ↳ vectors in n space $|\vec{a}||\vec{b}| \geq |\vec{a}\cdot\vec{b}|$

- L'Hôpitals rule
  ↳ $\lim_{x\to x_0}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to x_0}\left(\frac{f'(x)}{g'(x)}\right)$ where $f(x_0) = g(x_0) = 0$

- Limit doesnt exist if:
  ↳ blows up to ∞ ↳ approaches different values (discontinuous)
  ↳ infinite oscillations ↳ limit is at the end of an interval

- derivative from first principle
  $\frac{dy}{dx} = \lim_{\delta x\to 0}\left(\frac{f(x+\delta x) - f(x)}{\delta x}\right)$ limit for $\delta x \to 0^+$ and $\delta x \to 0^-$ must be equal

- product rule: $\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$ $\frac{d^n}{dx^n}fg = \sum_{r=0}^{n} {}^nC_r f^{(n-r)} g^r$

- quotient rule: $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - g'f}{g^2}$

- chain rule: $\frac{d}{dx}(f(g)) = f'(g(x))g'(x)$

- parametric differentiation: $\frac{d^2y}{dx^2} = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{(\dot{x})^3}$

- ellipse
  ↳ cartasean $x = a\cos t$ $y = b\sin t$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $b = a\sqrt{1-e^2}$

  ↳ polar $\frac{l}{r} = 1 + e\cos\theta$ $l = \frac{b^2}{a}$ $e = \sqrt{1 - \frac{b^2}{a^2}}$

  (alternatively $r = \frac{l}{1 - e\cos\theta}$)

- Riemanns definition of integration $A = \lim_{\delta x \to 0} \sum_{i=0} f(x_i)\,\delta x_i$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx \qquad \int_a^b f(x)dx = -\int_b^a f(x)dx$$

- Fundemental theorem of calculus $F(x) = \int_a^x f(u)du \qquad \frac{dF(x)}{dx} = f(x)$

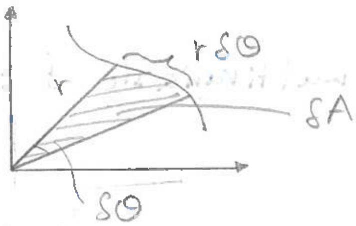
- infinite integrals : $\int_a^\infty f(x)dx = \lim_{n \to \infty} \int_a^n f(x)dx$

- improper integrals : $\int_0^1 x^{-\frac{1}{2}} dx = \lim_{\epsilon \to 0} \int_\epsilon^1 x^{-\frac{1}{2}} dx$

- mean: $\bar{f} = \frac{1}{b-a} \int_a^b f(x)dx$ - rms: $f_{rms} = \sqrt{\frac{1}{b-a} \int_a^b f^2(x)dx}$

- standard deviation : $\sigma = \sqrt{\frac{1}{b-a} \int_a^b (f-\bar{f})^2 dx} = \sqrt{f_{rms}^2 - \bar{f}^2}$

- area in polars: $A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 d\theta$



- path length

↳ parametric : $S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ cartesean: $S = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

↳ polar : $S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$

- centre of mass: $\bar{x} = \frac{\int_{x_1}^{x_2} xy\,dx}{\int_{x_1}^{x_2} y\,dx} \qquad \bar{y} = \frac{\frac{1}{2}\int_{x_1}^{x_2} y^2 dx}{\int_{x_1}^{x_2} y\,dx}$

~~↳ generally: $\bar{x}_i =$~~ $M = \rho A = \rho \int_{x_1}^{x_2} y\,dx$

- volume and SA of revolution

↳ about x axis: $SA = 2\pi \int_{x_1}^{x_2} y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$V = \pi \int_{x_1}^{x_2} y^2\, dx$$

$\left(\frac{\partial u}{\partial x}\right)_y$ - partial while keeping y constant

- partial derivatives

$\frac{\partial u}{\partial x} = \lim_{h \to 0} \left(\frac{u(x+h,y) - u(x,y)}{h} \right)$ $\quad \frac{\partial u}{\partial y} = \lim_{k \to 0} \left(\frac{u(x,y+k) - u(x,y)}{k} \right)$

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ if both are continuous

- differential of single variable f: $\quad df = \frac{df}{dx} dx$

- total differential for $u = u(x,y)$: $\quad du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

- chain rule for multivariable f: $\quad \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

↳ for $u = u(x,y)$, $x = x(r,s)$, $y = y(r,s)$ $\quad \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$ $\quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$

↳ when changing variables ensure all are changed

ensure same variable is being kept constant

$u(x,y) \equiv \tilde{u}(r,\theta)$ - different functional dependance

- partial differential opperators

$\frac{\partial u}{\partial x} = \frac{\partial \tilde{u}}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \tilde{u}}{\partial \theta}\frac{\partial \theta}{\partial x}$ $\quad \frac{\partial u}{\partial y} = \frac{\partial \tilde{u}}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \tilde{u}}{\partial \theta}\frac{\partial \theta}{\partial y}$

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \end{aligned}\right\}$$ partial diff operators relate rates of change in 2 coordinate systems

- Laplaces equation $\nabla^2 u = 0$

↳ cartesean $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ↳ polars $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{u}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \bar{u}}{\partial \theta^2} = 0$

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{u}}{\partial \theta^2}$$

- implicit differentiation of multivariable f

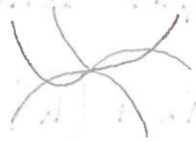
$$F = F(x,y) = 0 \qquad dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0 \implies \frac{dy}{dx} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial y}\right)} = -\frac{F_x}{F_y}$$

$$F = F(x,y,z) = 0 \qquad dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$$

constant x $\left(\frac{\partial z}{\partial y}\right)_x = -\frac{F_y}{F_z}$ constant y $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z}$ constant z $\left(\frac{\partial y}{\partial x}\right)_z = -\frac{F_x}{F_y}$

- 2D stationary points $\frac{\partial u}{\partial x} = 0$ $\frac{\partial u}{\partial y} = 0$ $(du = 0)$

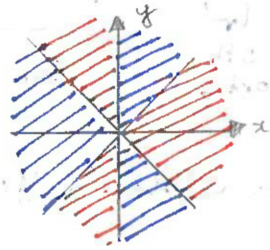
- Hessian matrix $H = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{pmatrix}$ $\det(H) = \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2$



$\det(H) < 0 \implies$ saddle point

$\det(H) > 0$: $\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_0 y_0} < 0 \implies$ local maximum

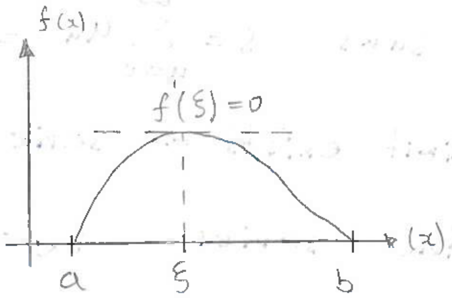
$\left.\frac{\partial^2 u}{\partial y^2}\right|_{x_0 y_0} > 0 \implies$ local minimum



- monkey saddle point

- Taylor series $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ about x_0

- Maclaurin series $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$



- Rolles theorem

↳ if $f(a) = f(b) = 0$ and $f(x)$ is continuous and differentiable between a and b.

there exists $a < \xi < b$

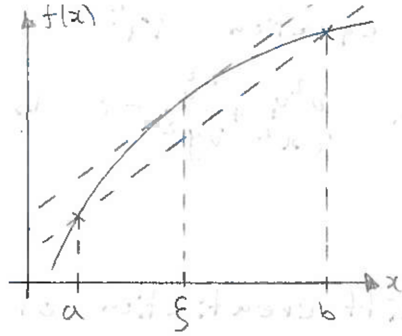
such that $f'(\xi) = 0$

- First mean value theorem

↳ consequence of Rolls theorem

there must exist ξ on a, b

such that $f'(\xi) = \dfrac{f(b) - f(a)}{b - a}$



- Taylors theorem

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \dots + f^{(n-1)}(x_0)\frac{(x-x_0)^{n-1}}{(n-1)!} + R_n$$

$$R_n = \frac{(x-x_0)^n}{n!} f^{(n)}(\xi_n) \qquad x_0 < \xi_n < x$$

if $\lim_{n \to \infty} R_n = 0$ series converges or terminates

- exponential (e.g.) e^x) allways wins over a polynomial

- double Taylor series $u(x,y)$ about $u(x_0, y_0)$

$$u(x,y) = \underbrace{u_0}_{\text{0th order}} + \underbrace{\left(h\left(\frac{\partial u}{\partial x}\right)_0 + k\left(\frac{\partial u}{\partial y}\right)_0\right)}_{\text{1st order}} + \underbrace{\frac{1}{2!}\left(h^2\left(\frac{\partial^2 u}{\partial x^2}\right)_0 + 2kh\left(\frac{\partial^2 u}{\partial x \partial y}\right)_0 + k^2\left(\frac{\partial^2 u}{\partial y^2}\right)_0\right)}_{\text{2nd order}} + \dots$$

$u(x,y) = u(x_0 + h, y_0 + k)$

using differential opperator $D \equiv h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$

$$u(x,y) = u_0 + Du_0 + \frac{D^2}{2!}u_0 + \frac{D^3}{3!}u_0 + \dots + \frac{D^n}{n!}u_0$$

- infinite sums $S = \sum_{n=0}^{\infty} u_n = \lim_{N \to \infty} \sum_{n=0}^{N} u_n$

↳ if limit exists ⇒ series converges

- geometric series $S_N = a\left(\frac{1 - r^{N+1}}{1 - r}\right)$ $\quad S_\infty = \frac{a}{1-r}$ $\quad u_n = ar^{n-1}$

- Harmonic series $u_n = \frac{1}{n}$ $\quad S_\infty = \sum_{n=1}^{\infty} \frac{1}{n}$

↳ diverges since as $\frac{1}{n}$ decreases the rate at which it decreases also decreases

- absolutely convergent

$S = \sum_{n=0}^{\infty} u_n$ absolutely convergent if $\sum_{n=0}^{\infty} |u_n|$ is also convergent

↳ absolute convergence implies regular convergence

- conditionally convergent

$S = \sum_{n=0}^{\infty} u_n$ conditionally convergent if $\sum_{n=0}^{\infty} u_n$ converges but

$\sum_{n=0}^{\infty} |u_n|$ does not converge

- $\lim_{n \to \infty} u_n = 0$ is necessary for convergence but not sufficient

- comparison test

↳ if u_n is given and a convergent series $\sum_{n=0}^{\infty} b_n$ exists

with a non-negative b_n such that $|u_n| \leq b_n$ for all n

then $\sum_{n=0}^{\infty} u_n$ is absolutely convergent

↳ similar for divergent series $\sum_{n=0}^{\infty} b_n$ $u_n \geq b_n$

- Leibnitz test - for alternating series

↳ if all 3 conditions true series will converge

1.) a_n is +ve 2.) a_n is decreasing $a_{n+1} < a_n$ 3.) $a_n \to 0$ as $n \to \infty$

for alternating series $\sum_{n=0}^{\infty} (-1)^n a_n$ $u_n = (-1)^n a_n$

- ratio test

for $\sum_{n=0}^{\infty} u_n$ $u_n \neq 0$ $L = \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right|$

↳ if $L < 1 \Rightarrow$ absolute convergence

if $L > 1 \Rightarrow$ divergence

if $L = 1$ not proven need other test

· radius of convergence for Taylor series

↳ ratio test used to determine for what values of x Taylor series converges set $L = 0$

e.g.) $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} u_n \quad u_n = \frac{x^n}{n!} \quad \left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n}\right| = \frac{|x|}{n+1}$

$L = \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$ true for all fixed $x \in \mathbb{C}$

so infinite radius of convergence

- Condition of integrability $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$

given a function $P(x,y)dx + Q(x,y)dy$