Complex Analysis 2021

- basic rules
  - ↳ commutative: $a+b = b+a \quad ab = ba$
  - ↳ associative: $a+(b+c) = (a+b)+c \quad a(bc) = (ab)c$
  - ↳ distributive: $a(b+c) = ab + ac$
  - ↳ unity 1 or $\mathbb{1}$ where $\frac{1}{a} a = \mathbb{1} \quad a \neq 0$

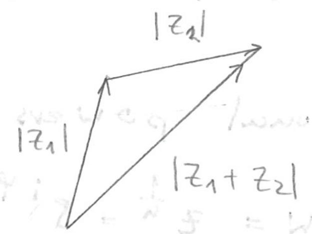
- introducing $\sqrt{-1} =: i$ doesn't break any rules

- complex number can be represented as a vector but we can multiply 2 complex numbers but not vectors

- complex conjugate $\quad z = x+iy \quad z^* = x - iy$

  $f(z^*) = (f(z))^* \quad (z \pm w')^* = z^* \pm w^* \quad (z\,w)^* = z^* w^* \quad \left(\frac{z}{w}\right)^* = \frac{z^*}{w^*}$

- absolute value $\quad |z| = \sqrt{z z^*} = \sqrt{x^2+y^2} \quad |zw| = |z||w|$
  - ↳ careful when using absolute values in inequalities since $|z| > |w|$ would imply $-5 > -3$



- triangle inequality $\quad |z_1 + z_2| \leq |z_1| + |z_2|$
  - ↳ proof since $|z_1 + z_2|^2 = |z_1| + |z_2| + 2\,\mathrm{Re}(z_1 z_2^*)$

- polar coordinates

  $r = |z| = \sqrt{x^2+y^2} \quad \theta = \arg(z) = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & x>0 \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & x<0 \end{cases}$

- multiplying by i rotates by $90°$ anti-clockwise

Eulers formula $e^{i\theta} = \cos\theta + i\sin\theta$

$|z_1 z_2| = |z_1||z_2|$ $\quad \arg(\theta_1 + \theta_2) = \arg(\theta_1) + \arg(\theta_2)$

$$|z_1|e^{i\theta_1}\,|z_2|e^{i\theta_2} = |z_1||z_2|\,e^{i(\theta_1+\theta_2)}$$

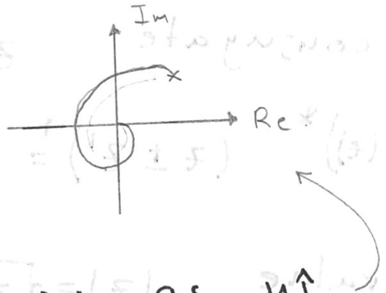
De Moivres theorem $\quad (\cos\theta + i\sin\theta)^n \equiv \cos(n\theta) + i\sin(n\theta)$

can apply Binomial expansion $\quad \cos(n\theta) + i\sin(n\theta) = \sum_{k=0}^{n} {}^nC_k (i)^{n-k} \cos^k(\theta)\sin^{n-k}(\theta)$

roots of unity $\quad 1 = e^{i2\pi k} \quad k \in \mathbb{Z}$

nth root $\quad (e^{i2\pi k})^{\frac{1}{n}} = e^{\frac{i2\pi k}{n}} \quad k \in \mathbb{Z} \quad k \leq n-1$ $\quad$ since at $k = n$ roots repeat

- powers of complex numbers

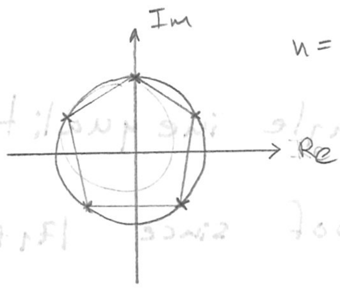


$z = re^{i\theta} \quad z^n = r^n e^{in\theta} \quad n \in \mathbb{N}$

↳ for $r = |z| < 1$ $\quad z^n$ spirals towards origin as $n \uparrow$

for $r = |z| > 1$ $\quad z^n$ spirals outwards to ∞ as $n\uparrow$

- fractional powers (roots) $\quad \sqrt[n]{z} = (z)^{\frac{1}{n}}$



let $w = z^{\frac{1}{n}} = \rho^{i\varphi}$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\frac{\theta + 2\pi k}{n}} \quad \{k \in \mathbb{Z} \mid k < n\}$$

↳ n equidistant points on circumference of circle radius $r^{\frac{1}{n}}$ giving regular polygon

- rational powers $\quad z^{\frac{p}{q}} = (z^{\frac{1}{q}})^p$

- irrational powers $a \in \mathbb{I}$
  ↳ ∞ no of powers since they don't loop onto eachother (repeat)

$$z^a = r^a (e^{i\theta})^a = r^a e^{ia(\theta + 2\pi k)} \quad k \in \mathbb{Z} \quad (k \text{ goes to } \infty)$$

- complex powers $w \in \mathbb{C}$ $w = a + ib$

$r^w = r^{a+ib}$ can be expressed in the form $\rho e^{i\varphi}$

$$(e^{i\theta})^w = e^{i(\theta + 2\pi k)(a+ib)}$$

$$= \underbrace{e^{-b(\theta+2\pi k)}}_{\mathbb{R}} \underbrace{e^{ia(\theta + 2\pi k)}}_{\mathbb{C}} \quad k \in \mathbb{Z} \quad (k \text{ goes to } \infty)$$

↳ again ∞ no of powers since they don't repeat

- function can be expressed as
  ↳ power series
  ↳ integral representation
  ↳ differential equation

- ratio test for convergence series $\sum_{n=0}^{\infty} A_n$ where $A_n = a_n x^n$

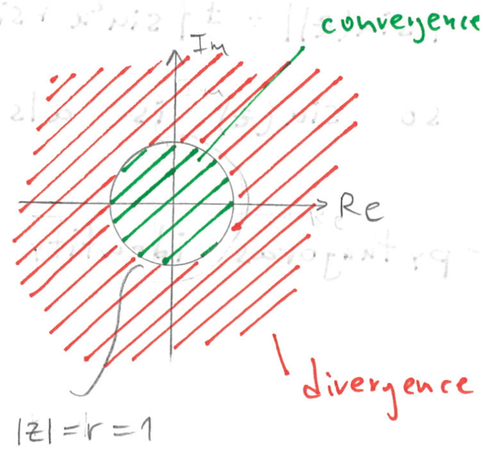
define $\rho = \lim_{n \to \infty} \left| \frac{A_{n+1}}{A_n} \right|$ $\quad A_n \neq 0$

↳ if $\rho < 1 \Rightarrow$ absolute convergence
  $\rho > 1 \Rightarrow$ divergence
  $\rho = 1 \Rightarrow$ undefined so ratio test doesn't work

↳ eg.) $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \dots$

$\rho = \lim_{n \to \infty} \left| \frac{z^{n+1}}{z^n} \right| = |z|$ $\quad$ for convergence $\rho = |z| < 1$

so if $|z| < 1$ $\sum_{n=0}^{\infty} z^n$ converges



- Cauchy product

two series $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \dots$ and $\sum_{n=0}^{\infty} b_n = b_0 + b_1 + \dots$

find product $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = a_0 b_0 + (a_0 b_1 + b_0 a_1) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \dots$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)$$

↳ so product of 2 infinite sums can be written as an infinite sum of finite sums

↳ used to proove $e^{z_1} e^{z_2} = e^{z_1+z_2}$

$$e^{z_1} e^{z_2} = \left(\sum_{n=0}^{\infty} \frac{z_1^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{z_2^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{(z_1+z_2)^2}{n!} = e^{z_1+z_2}$$

complex trigonometric functions

$\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ $\sin(\theta) = \frac{1}{2}(e^{i\theta} - e^{-i\theta})$

$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$

$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

- bounds of complex trig functions

$|\cos(z)| = \pm\sqrt{\cos^2 x + \sinh^2 y}$

as $y \to \infty$ $\sinh y \to \infty$ so $\cos(z)$ is unbounded as $\mathrm{Im}(z) \to \infty$

$|\sin(z)| = \pm\sqrt{\sin^2 x + \sinh^2 y}$

so $\sin(z)$ is also unbonded as $\mathrm{Im}(z) \to \infty$

- pythagoras identity $\cos^2(z) + \sin^2(z) = 1$ still holds

- zeros of complex trig functions

$\sin(z) = 0 \Rightarrow \sin^2(z) = 0$ $\quad$ $\sinh^2 y = 0$ only if $y=0$

$\sin^2 x + \sinh^2 y = 0$ $\quad$ so $y = 0$ so $z \in \mathbb{R}$

$\cos(z) = 0 \Rightarrow \cos^2(z) = 0$ $\quad$ $\sinh^2 y = 0$ so again $z \in \mathbb{R}$

$\cos^2 x + \sinh^2 y = 0$

↳ all zeros of complex trig functions are real

- periodicity of complex trig functions

↳ complex trig functions are still periodic in $2\pi n$ $\quad n \in \mathbb{Z}$

$\cos(z + 2\pi n) = \cos(z)$ $\qquad$ $\sin(z + 2\pi n) = \sin(z)$

- logarithm of a complex number

$z = re^{i\theta} = re^{i(\theta + 2\pi n)}$ $\quad n \in \mathbb{Z}$

so $\ln(z) = \ln(re^{i(\theta + 2\pi n)}) = \ln(r) + i(\theta + 2\pi n)$

this would give ∞ no of solutions so restrict θ

$\ln(z) = \ln(r) + i\theta \quad -\pi \leq \theta < \pi$

- logarithm of a -ve number

$\ln(-|x|) = \ln(e^{-i\pi}|x|) = \ln(|x|) - i\pi$

- inverse of complex trig functions

$\sin^{-1}(z) = -i\ln(iz \pm \sqrt{1 - z^2})$

$\cos^{-1}(z) = -i\ln(z \pm i\sqrt{1 - z^2})$

↳ these would be multivalued functions if we don't restrict the logarithm

- definition of continuity
  - function $f(z)$ is continuous at $f(z_0)$ if $\lim\limits_{z \to z_0} f(z) = f(z_0)$

    so limit must exist and $f(z)$ must be defined at z_0
  - epsilon-delta definition:

    for each $\varepsilon > 0$ there is a $\delta > 0$ (dependent on ε) such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$

- properties of continuous functions
  - given that $f(z)$ and $g(z)$ are continuous at z_0
    - sum $f(z) + g(z)$ is also
    - product $f(z)\, g(z)$ is also
    - composition $g(f(z))$ is also

- in practice use $\lim\limits_{z \to z_0} f(z) = \lim\limits_{\delta \to 0} f(z_0 + \delta)$

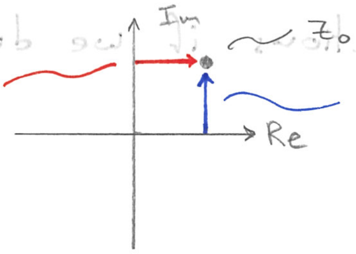
- derivative wrt complex number

$$\frac{df}{dz} = \lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z} \qquad z, \delta z \in \mathbb{C}$$

  - if $f(z)$ has a derivative $\frac{df}{dz}$ in a region of complex plane R then $f(z)$ is analytic in R
  - derivative exists if the limit exists

- in complex plane we have to consider directionality so direction from which we approach limit

approaching along real axis $\delta z = \delta z^*$



approaching along imaginary axis $\delta z = -\delta z^*$

- example $f(z) = |z|^2$

$$\frac{\delta f}{\delta z} = \frac{|z+\delta z|^2 + |z|^2}{\delta z} = \frac{(z+\delta z)(z^* + \delta z^*) - zz^*}{\delta z} = z^* + \delta z^* + z\frac{\delta z^*}{\delta z}$$

↳ if $z = 0$ $z^* = 0$ $\frac{df}{dz} = \lim_{\delta z \to 0} \frac{\delta f}{\delta z} = 0$

↳ if $z \neq 0$

for $\delta z \in \mathbb{R}$ $\delta z = \delta z^*$ $\frac{df}{dz} = \lim_{\delta z \to 0}\left(z^* + \delta z^* + z\frac{\delta z^*}{\delta z}\right) = z^* + z$

for $\delta z \subset \mathbb{I}$ $\delta z = -\delta z^*$ $\frac{df}{dz} = \lim_{\delta z \to 0}\left(z^* + \delta z^* + z\frac{\delta z^*}{\delta z}\right) = z^* - z$

since the two limits are different there is no limit and $f(z)$ is only analytic at $z=0$

- Cauchy - Riemann equations

↳ for function $f(z) = u(x,y) + i\upsilon(x,y)$

calculate limits for $\delta z = \delta z^* \Rightarrow \delta z = x$ and $\delta z = -\delta z^* \Rightarrow \delta z = i\delta y$

then equate them

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial \upsilon}{\partial y}(x_0, y_0) \qquad \frac{\partial \upsilon}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0)$$

↳ any analytic function has continuous partial derivatives and satisfies Cauchy - Riemann equations

- holomorphic function

↳ complex valued function which is complex differentiable in the neighbourhood of each point in the complex domai[n]

- existance and uniqueness theorem for 1st order ODE

↳ for $f(x,y) = y'$ if $f(x,y)$ is continuous on some region R $\{(x,y): |x - x_0| \leq a, |y - y_0| \leq b\}$ and so is $\frac{\partial f}{\partial y}$ then there exists a unique solution to $y' = f(x,y)$

- separable 1st order ODE

$$F(x, y, \frac{dy}{dx}) = 0 \quad \text{so} \quad \frac{dy}{dx} = f(x,y) \Rightarrow \frac{dy}{dx} = f(y)\, g(x)$$

$$\int \frac{1}{f(y)}\, dy = \int g(x)\, dx + C$$

- linear DE - has 0th and 1st order derivatives $(y, \frac{dy}{dx})$

- homogeneous DE - has right hand side $= 0$

- linear homogeneous 1st order ODE

$$\frac{dy}{dx} + p(x)y = 0 \Rightarrow \int \frac{1}{y}\, dy = -\int p(x)\, dx + C$$

- linear inhomogeneous 1st order ODE

$\frac{dy}{dx} + p(x)y = q(x)$ integrating factor $R = e^{\int p(x)\, dx}$ (omit $+C$)

then solve $\frac{d}{dx}(Ry) = R\,q(x)$

$$y(x) = e^{-\int p(x)\, dx}\left(\int e^{\int p(x)\, dx} q(x)\, dx + C\right)$$

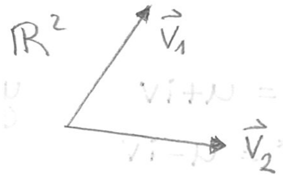
- superposition principle for 2nd order ODE
  - ↳ if $y_1(x)$ and $y_2(x)$ are solutions then $y = Ay_1(x) + By_2(x)$ must also be a solution
  - ↳ A, B can be varied continuously to get ∞ of solutions

- existance and uniqueness for 2nd order ODE

$a(x)\, y'' + b(x)\, y' + c(x)\, y = 0 \qquad IC: y(x_0) = y_0 \quad y'(x_0) = y_0'$

↳ has a unique solution if $a(x), b(x), c(x)$ are continuous and $a(x) \neq 0$. for some given IC

- general solution is $y(x) = A y_1(x) + B y_2(x)$
where $y_1(x)$ and $y_2(x)$ are linearly independent
so $y_1(x) \neq \lambda y_2(x)$



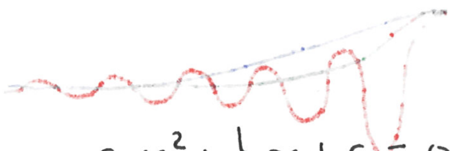
- with vectors $\vec{v}_1$ and $\vec{v}_2$ are linearly independent if they are at different angles
so any vector can be written as $\vec{v} = \alpha \vec{V}_1 + \beta \vec{V}_2$

- Wronskian $W(y_1, y_2)$
↳ used to test linear dependancy

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix} = y_1(x) \frac{dy_2}{dx} - \frac{dy_1}{dx} y_2(x)$$

↳ if y_1 and y_2 are analytical and $W(y_1, y_2) = 0$
then y_1 and y_2 are linearly independant

- solving 2nd order ODE (homogeneous)



$ay''(x) + by'(x) + cy(x) = 0 \implies$ characteristic eq $a\alpha^2 + b\alpha + c = 0$
using trial solution $y = e^{\alpha x}$ $y' = \alpha e^{\alpha x}$ $y'' = \alpha^2 e^{\alpha x}$

↳ case 1: $\alpha_1 \neq \alpha_2$ and $\alpha_1, \alpha_2 \in \mathbb{R}$

$$y(x) = Ae^{\alpha_1 x} + Be^{\alpha_2 x} \qquad A = \frac{\alpha_2}{\alpha_2 - \alpha_1}(y_0 - y_0') \qquad B = \frac{\alpha_1}{\alpha_1 - \alpha_2}(y_0 - y_0')$$

↳ case 2: $\alpha_1 = \alpha_2$ and $\alpha_1 = \alpha_2 \in \mathbb{R}$ degenerate solution
↳ $b^2 - 4ac = 0$
↳ $\alpha_1 = \alpha_2$ so $\alpha_2 - \alpha_1$ in denominator $= 0$ but also $e^{\alpha_1 x}$ and $e^{\alpha_2 x}$ substracted give 0 so use L'Hôpital

$$y(x) = (A + Bx)e^{\alpha x} \qquad A = y_0 \qquad B = y_0' - \alpha y_0$$

↳ case 3: $\alpha_1 = \alpha_2^*$ $\alpha_1, \alpha_2 \in \mathbb{C}$ $b^2 - 4ac < 0$

$$y(x) = A e^{\alpha x} + B e^{\alpha^* x}$$

let $\alpha = u + iv$ $y(x) = e^{ux}\left(C\cos(vx) + D\sin(vx)\right)$ $C = A + B$
$\alpha^* = u - iv$ $D = i(A - B)$

or $y(x) = A_0 e^{ux}\cos(vx + \phi)$ $A_0 = \sqrt{C^2 + D^2} = 2\sqrt{AB}$

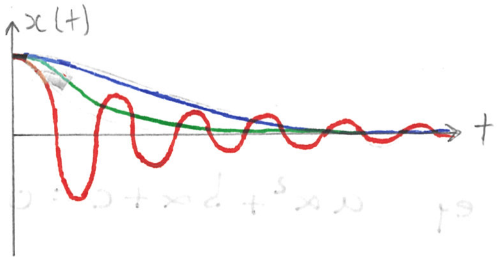
$\phi = \tan^{-1}\left(-\frac{D}{C}\right)$

damping $\gamma = \frac{b}{m}$ $\omega_0^2 = \frac{k}{m}$

↳ Case 1: $\gamma^2 - 4\omega_0^2 > 0$ overdamped solution $\in \mathbb{R}$
so decays exponentially

↳ Case 2: $\gamma^2 - 4\omega_0^2 = 0$ critical solution $\in \mathbb{R}$ and degenerate
so exponential decay (fastest)

↳ Case 3: $\gamma^2 - 4\omega_0^2 < 0$ underdamped solution $\in \mathbb{C}$
so oscillatory behaviour which decays



- Solving 2nd order ODE (inhomogeneous)

$$ay'' + by' + cy = f(x)$$

↳ principle of superposition only holds for homogeneous ODE

↳ general solution for inhomogeneous = general solution for homogeneous + particular solution

↳ particular solution found via method of undetermined coefficients

example trial functions

polynomial: $y(x) = ax + b$ trig: $y(x) = a\sin(px) + b\cos(px)$ exp: $y(x) = ae^{px}$

- solving system of coupled ODE

↳ example:

$$m \frac{d\vec{v}}{dt} = q\vec{V} \times \vec{B} \quad \vec{B} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} \Rightarrow m \frac{dV_x}{dt} = qV_y B \quad m \frac{dV_y}{dt} = -qV_x B$$

introduce complex velocity $V = V_x + iV_y$

$$w_c = \frac{qB}{m}$$

$$\frac{dV}{dt} = \frac{dV_x}{dt} + i\frac{dV_y}{dt} = \frac{1}{m}\left(qV_yB + i(-q)V_xB\right) = -iw_cV$$

↳ cyclotron frequency

$$\frac{dV}{dt} = -iw_cV \qquad V(t) = V_0 e^{-w_ct} \qquad V_x(t) = Re(V(t)) \quad V_y(t) = Im(V(t))$$

IC: $V_0 = V_{x_0} + iV_{y_0}$

- Abels identity

↳ for a 2nd order homogeneous ODE with non-constant coef.

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{Wronskian } W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix} = y_1\frac{dy_2}{dx} - \frac{dy_1}{dt}$$

$$W(y_1, y_2) = Ce^{-\int p(x)dx}$$ where C is a constant

↳ this can be generalised to nth order linear homogenous OD

- a set of initial conditions provided must be linearly-independant