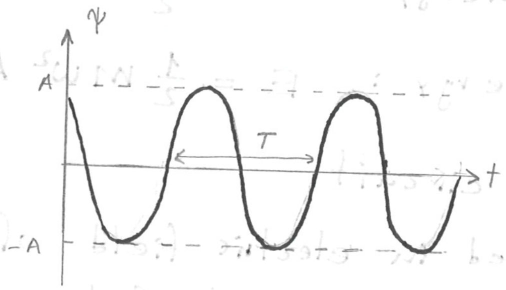
Oscillations and Waves 2021

- simple harmonic motion
  - ↳ restoring force is directly proportional and opposite to displ

$$\Psi(t) = A\cos(\omega t + \phi)$$
$$= B\cos(\omega t) + C\sin(\omega t)$$

$$B = A\cos(\phi) \quad C = -A\sin(\phi)$$

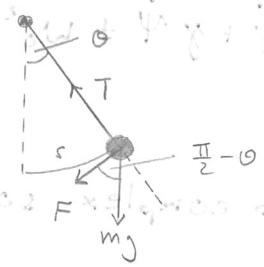


- Hooke's law
  - ↳ for small displacements restoring force is $F = -kx$
  - ↳ usually valid for any system with equilibrium and small displacement

- pendulum

use SAA $\sin\theta \approx \theta$

$$F = -mg\cos\left(\frac{\pi}{2} - \theta\right) \qquad F = -mg\theta$$
$$= -mg\sin\theta$$



- SHM as solution to Hooke's law

$$m\frac{d^2\Psi}{dt^2} = -k\Psi \Rightarrow \Psi = A\cos(\omega t + \phi) \quad \text{where} \quad \omega^2 = \frac{k}{m}$$

↳ for pendulum $k = \frac{mg}{\ell}$ $\quad \omega = \sqrt{\frac{g}{\ell}}$

- complex variables
  - ↳ introduce complex displacement $\tilde{\Psi}(t)$ and amplitude $\tilde{A} = Ae^{i\phi}$

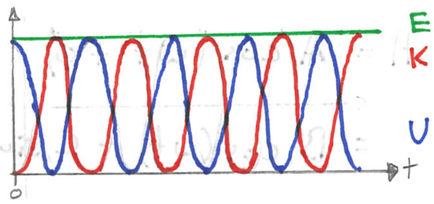
$$\tilde{\Psi}(t) = \tilde{A}e^{i\omega t} \qquad \dot{\tilde{\Psi}} = i\omega\tilde{A}e^{i\omega t} = i\omega\tilde{\Psi}$$
$$= Ae^{i\phi}e^{i\omega t} \qquad \ddot{\tilde{\Psi}} = -\omega^2\tilde{A}e^{i\omega t} = -\omega^2\tilde{\Psi}$$
$$= A\left(\cos(\omega t + \phi) + i\sin(\omega t + \phi)\right)$$

· energy of undamped SHM

potential energy : $U = \frac{1}{2} k \psi^2 = \frac{1}{2} m \omega^2 A^2 \cos^2(\omega t + \phi)$

kinetic energy : $K = \frac{1}{2} m \dot{\psi}^2 = \frac{1}{2} m \omega^2 A^2 \sin^2(\omega t + \phi)$

total energy : $E = \frac{1}{2} m \omega^2 A^2$



↳ for LC circuit

$U \Rightarrow$ stored in electric field of capacitor

$K \Rightarrow$ stored in magnetic field of inductor

· damped harmonic motion DHM

↳ introduce damping force $F_d = -b\dot{\psi}$

$\ddot{\psi} + \gamma \dot{\psi} + \omega_0^2 \psi = 0$ damping constant $\gamma = \frac{b}{m}$

natural frequency $\omega_0^2 = \frac{k}{m}$

↳ complex solution $\tilde{\psi}(t) = \tilde{A} e^{i\omega t}$

characteristic equation $\omega^2 - i\omega\gamma - \omega_0^2 = 0 \Rightarrow \omega = \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$

↳ define quality factor which characterises how underdamped the system is $Q = \frac{\omega_0}{\gamma} = \sqrt{\frac{km}{b^2}}$

- heavy damping $Q < \frac{1}{2}$ $\omega \in \mathbb{C}$

↳ $i\omega t$ is real so exponential decay and also $\tilde{\psi}$ is real

- critical damping $Q = \frac{1}{2}$ $\omega = \frac{i\gamma}{2}$ since $\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} = 0$

$\tilde{\psi} = \tilde{A} e^{-\frac{\gamma}{2} t}$ $\psi(t) = (A + Bt) e^{-\frac{\gamma}{2} t}$

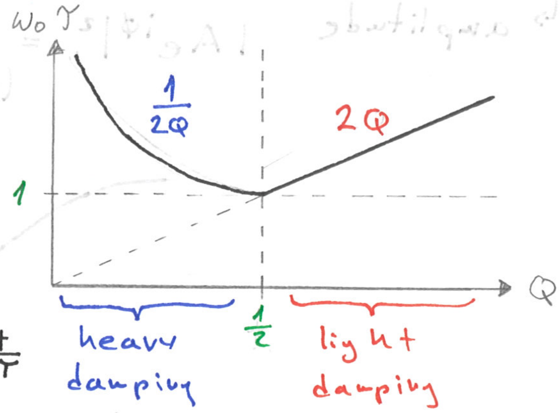
↳ $\dot{\psi} = -\frac{\gamma}{2}\psi$ so ψ and $\dot{\psi}$ are not linearly independant

so IC can not be specified as ψ_0 and $\dot{\psi}_0$

↳ fastest way for system to return to equilibrium

- light damping $Q > \frac{1}{2}$ $\omega \in \mathbb{R}$
  - $i\omega t$ is complex $i\omega t = -\frac{\gamma}{2}t \pm i\omega_d t$ $\omega_d = \sqrt{\omega_0^2 - (\frac{\gamma}{2})^2}$

    so system oscillates at ω_d. $\omega_d < \omega_0$
  - amplitude decays under envelope function $e^{-\frac{\gamma}{2}t}$



- Time taken to reach equilibrium
  - introduce decay time Υ which is the time taken for amplitude to fall by factor of e
  - light damping $\Upsilon = \frac{2}{\gamma}$ $A \propto e^{-\frac{\gamma}{2}t} = e^{-\frac{t}{\Upsilon}}$
  - heavy damping $\tilde{\Upsilon} = \frac{\gamma}{2\omega_0^2}$ $\tilde{\Upsilon} = \frac{\Upsilon_1 + \Upsilon_2}{2}$ $\psi(t) = Be^{-\frac{t}{\Upsilon_1}} + Ce^{-\frac{t}{\Upsilon_2}}$

- forced harmonic motion FHM
  - apply driving force $F = F_0 \cos(\omega t)$ ω - driving freq
  - $\ddot{\psi} + \gamma\dot{\psi} + \omega_0^2\psi = \frac{F_0}{m}\cos(\omega t) \Rightarrow \ddot{\tilde{\psi}} + \gamma\dot{\tilde{\psi}} + \omega_0^2\tilde{\psi} = \frac{F_0}{m}e^{i\omega t}$
  - solution composed of steady state + transient

- Steady state
  - solution when E supplied by driver = E lost by damping so system oscillates at ω with constant amplitude
  - frequency of oscillation = driving frequency (can factor out $e^{i\omega t}$)

define : $A_0 = \frac{F_0}{k}$ $Q = \frac{\omega_0}{\gamma}$ $W = \frac{\omega}{\omega_0}$ ϕ - phase difference

$\tilde{\psi}(t) = \tilde{A}e^{i\omega t}$ $\tilde{A} = \frac{A_0}{1 - W^2 + \frac{iW}{Q}}$ (all W are W)

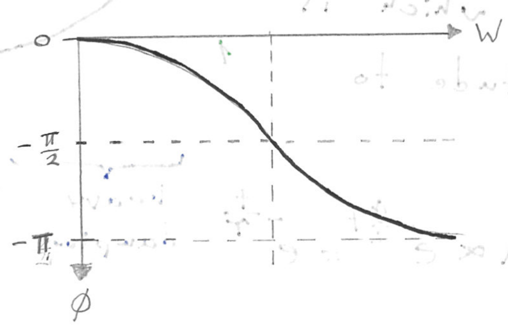
- steady state is the particular solution for a given driving force

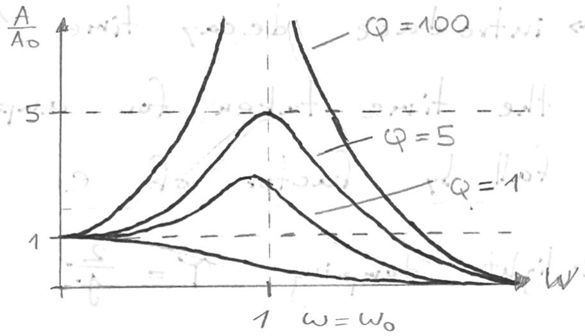
- resonance

↳ since $W = \frac{\omega}{\omega_0} > 0$ we can restrict $-\pi \le \phi \le 0$

$$\tan\phi = -\frac{\frac{W}{Q}}{(1-W^2) - i\frac{W}{Q}} = \frac{W}{iW - Q(1-W^2)}$$

↳ amplitude $|Ae^{i\phi}|^2 = \frac{A_0^2}{(1-W)^2 + \frac{W^2}{Q^2}}$ $\quad A = |Ae^{i\phi}| = \frac{A_0}{\sqrt{(1-W)^2 + \frac{W^2}{Q^2}}}$





$W = 0 \quad \omega = 0 \quad \tan\phi = 0 \quad \phi = 0 \quad A = A_0$

$W = 1 \quad \omega = \omega_0 \quad \tan\phi \to \infty \quad \phi = -\frac{\pi}{2} \quad A = QA_0$

$W \to \infty \quad \omega \to \infty \quad \tan\phi \to 0 \quad \phi \to -\pi \quad A \to 0$

$\left(\frac{1}{QW}\right)$

↳ resonance occours at ω is close to natural frequency ω_0

$$\omega_{res} = \omega_0\sqrt{1 - \frac{1}{2Q^2}} \qquad \text{as } Q \to \infty \quad \omega_{res} \to \omega$$

- maximum velocity

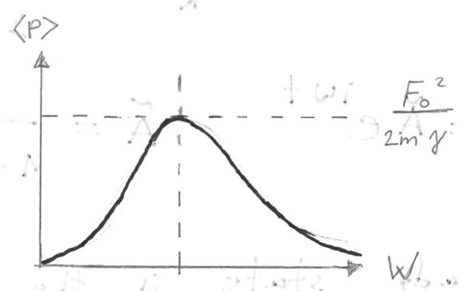
↳ velocity is maximum exactly at $W = 1$ so $\omega_{res} = \omega$

- time average power over 1 cycle

$$\frac{1}{T}\int_0^T \sin^2(\omega t)\,dt = \frac{1}{2}$$

$$P(t) = b\omega^2 A^2 \sin^2(\omega t + \phi) = -b\dot{\psi}^2$$

$$\langle P\rangle = \frac{F_0^2}{2Qm\omega_0\left(\left(\frac{1}{W} - W\right)^2 + \frac{1}{Q^2}\right)}$$



$\langle P\rangle_{max}$ at $W = 1 \qquad \langle P\rangle_{max} = \frac{F_0^2}{2m\gamma}$

$W \ll 1 \quad \langle P\rangle \propto W^2$

$W \gg 1 \quad \langle P\rangle \propto \frac{1}{W^2}$

- transient

↳ the transient is the solution to the homogenous equatio

so set RHS to 0 $\ddot{\tilde{\Psi}} + \gamma \dot{\tilde{\Psi}} + \omega_0^2 \tilde{\Psi} = 0$

- general solution

↳ general solution is a combination of transient + steady stat

steady state: $\Psi_1(t) = A_1 \cos(\omega t + \phi_1)$

transient: $\Psi_2(t) = \mathrm{Re}(A_2 e^{i\phi_2} e^{i\omega_2 t})$ ← solution to DHM

$\Psi(t) = \Psi_1(t) + \Psi_2(t)$ $\quad \underbrace{\ddot{\Psi}_1 + \gamma \dot{\Psi}_1 + \omega_0^2 \Psi_1}_{\text{steady state}} + \underbrace{\ddot{\Psi}_2 + \gamma \dot{\Psi}_2 + \omega_0^2 \Psi_2}_{\text{transient } = 0} = \frac{F_0}{m}\cos(\omega t)$

↳ system transitions from initial state to steady state via decay of transient

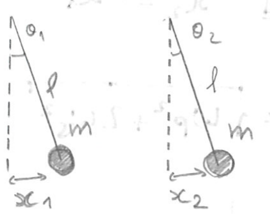
↳ transient depends on IC, steady state depends on $F_{driving}$

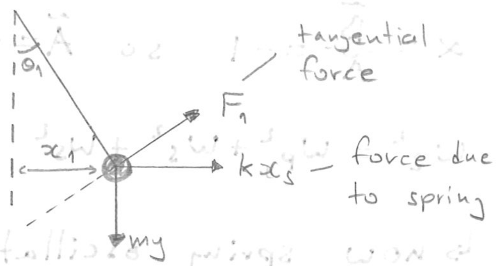
- normal modes NM

↳ used for analysis of oscillation of complex systems

↳ states of motion in which all parts of system oscillate at the same freq

- coupled pendulums





$F_1 = -mg\sin\theta_1 + kx_s\cos\theta_1$

$x_1 = l\sin\theta_1 \approx l\theta_1$

$F_2 = -mg\sin\theta_2 - kx_s\cos\theta_2$ $\qquad x_s = x_2 - x_1$

$x_2 = l\sin\theta_2 \approx l\theta_2$

use SAA $\sin\theta \approx \theta$ $\quad \cos\theta \approx 1$

$F_1 = -mg\frac{x_1}{l} + k(x_2 - x_1)$ $\qquad F_2 = -mg\frac{x_2}{l} - k(x_2 - x_1)$

to get equations of motion $F_1 = m\ddot{x}_1$ $\quad F_2 = m\ddot{x}_2$

↳ $\omega_p = \sqrt{\frac{g}{l}}$ freq of pendulum if there were no spring

$\omega_s = \sqrt{\frac{k}{m}}$ freq of spring if there were no pendulum

these give equations of motion

$$m\ddot{x}_1 = -m\frac{g}{l}x_1 + k(x_2 - x_1) \Rightarrow \ddot{x}_1 = -\omega_p^2 x_1 + \omega_s^2(x_2 - x_1)$$

$$m\ddot{x}_2 = -m\frac{g}{l}x_2 - k(x_2 - x_1) \Rightarrow \ddot{x}_2 = -\omega_p^2 x_2 - \omega_s^2(x_2 - x_1)$$

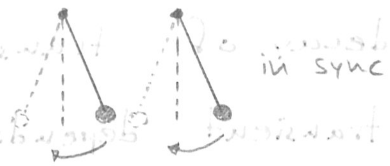
↳ assume the solutions to be: $\tilde{x}_1(t) = \tilde{A}e^{i\omega t}$ $\tilde{x}_2(t) = \tilde{B}e^{i\omega t}$

normal modes must satisfy $\omega^2 = \omega_p^2 - \omega_s^2\alpha + \omega_s^2$

$\left(\alpha = \frac{\tilde{B}}{\tilde{A}}\right)$ $\omega^2 = \omega_p^2 - \omega_s^2\frac{1}{\alpha} + \omega_s^2$

these give normal modes ω_1 and ω_2 at $\alpha = +1, -1$

- mode 1 $\alpha = +1$



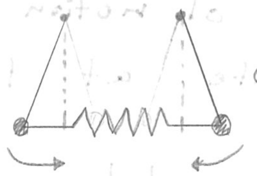
$\alpha = \frac{\tilde{B}}{\tilde{A}} = 1$ so $\tilde{A} = \tilde{B}$ and $\tilde{x}_2 = \tilde{x}_1$

$\left(\tilde{x}_1 = \tilde{x}_2 = A\cos(\omega_p t + \phi_1)\right)$

$\omega_1^2 = \omega_p^2 - \omega_s^2 + \omega_s^2 \Rightarrow \omega_1 = \omega_p$

↳ so spring doesn't oscillate and pendulums move in sync

- mode 2 $\alpha = -1$



$\alpha = \frac{\tilde{B}}{\tilde{A}} = -1$ so $\tilde{A} = -\tilde{B}$ and $\tilde{x}_2 = -\tilde{x}_1$

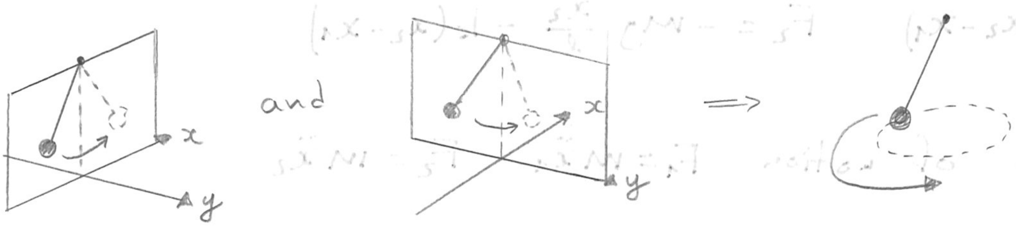
$\omega_2^2 = \omega_p^2 + \omega_s^2 + \omega_s^2 \Rightarrow \omega_2 = \sqrt{\omega_p^2 + 2\omega_s^2}$ $\left(\tilde{x}_1 = -\tilde{x}_2 = A\cos(\omega_2 t + \phi_2)\right)$

↳ now spring oscillates, pendulums are out of phase by 180°

and ω_2 depends on both ω_s and ω_p

- normal modes of 3D pendulum

↳ swing in 2 planes which can be superposed



- superposing normal modes for double pendulum

IC: $x_1(t=0) = x_0 \quad x_2(t=0) = 0 \quad \dot{x}_1(t=0) = 0 \quad \dot{x}_2(t=0) = 0$

from IC: $x_1(t) = \frac{x_0}{2}\cos(\omega_1 t) + \frac{x_0}{2}\cos(\omega_2 t)$

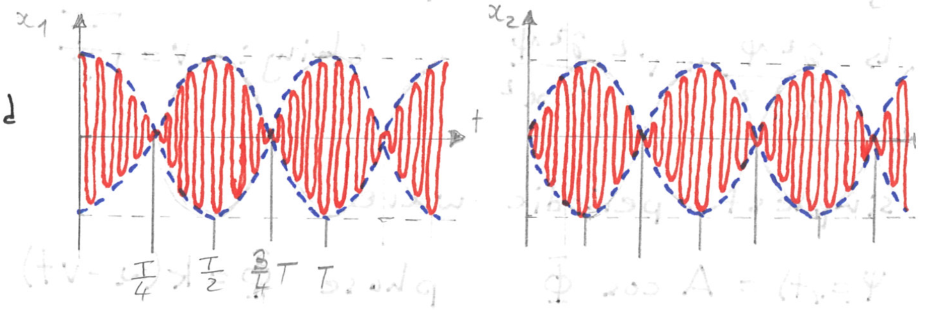
$$x_2(t) = \frac{x_0}{2}\cos(\omega_1 t) - \frac{x_0}{2}\cos(\omega_2 t)$$

use identity: $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

overall: $x_1(t) = x_0\cos(\omega_h t)\cos(\omega_l t) \quad x_2(t) = x_0\sin(\omega_h t)\sin(\omega_l t)$

↳ so masses oscillate at high freq ω_h modulated by low freq envelope ω_l



↳ energy exchanged every $\frac{T}{4}$

- wave

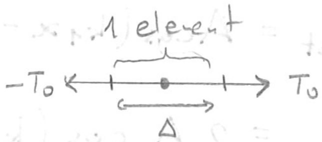
↳ disturbance which travels in certain direction with no net transport of mass

↳ any function $f(x-vt)$ represents a wave moving right

↳ waves do not have to be periodic

- waves on a stretched string

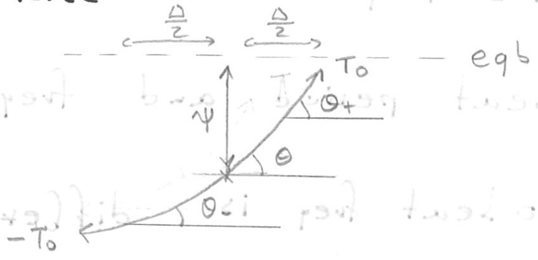
↳ without disturbance string is stationary along x axis with tension T_0 and mass per length S_0



↳ since string element Δ is already stretched assume change in tension to be negligable $T = T_0$

↳ using taylor expansion for Θ restoring force due to tension is

$$F = T_0 \Delta \frac{\partial \Theta}{\partial x}$$



↳ gradient $\frac{\partial \psi}{\partial x} = \tan\Theta \approx \Theta$

$$F = T_0 \Delta \frac{\partial^2 \psi}{\partial x^2}$$

↳ since mass of element $m = \rho_0 \Delta$ $\quad \frac{\partial^2 \psi}{\partial t^2} = \frac{T_0}{\rho_0} \frac{\partial^2 \psi}{\partial x^2}$

- Sound waves

↳ derivation is beyond scope of syllabus but I have it in further notes

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\gamma P_0}{\rho_0} \frac{\partial^2 \psi}{\partial x^2}$$

ρ_0 - density $\quad P_0$ - pressure

γ - ratio of specific heats

- comes from adiabetic law

for air $\gamma = 1.4$

- general wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

string: $v = \sqrt{\frac{T}{\rho}}$ $\quad$ sound $V = \sqrt{\frac{\gamma P}{\rho}}$

- simplest periodic wave

$\Psi(x,t) = A \cos \Phi$ $\quad$ phase $\Phi = k(x - vt)$ $\quad kV = \omega$ $\quad V = \frac{\omega}{k}$

- general monochromatic sinusodial wave

$\tilde{\Psi}(x,t) = \tilde{A} e^{i(kx - \omega t)} = A e^{i\phi} e^{i(kx - \omega t)}$ $\quad$ $k > 0$ $\;+x$ direction, $\;k < 0$ $\;-x$ direction

$\Psi(x,t) = \mathrm{Re}(\tilde{\Psi}(x,t)) = A \cos(kx - \omega t + \phi)$

- superposition of waves with same A and velocity

↳ same amplitude A same velocity so $\frac{\omega_1}{k_1} = \frac{\omega_2}{k_2} = V$

$\Psi_{tot} = A\cos(k_1 x - \omega_1 t) + A \cos(k_2 x - \omega_2 t)$

$= 2A \cos(k_h x - \omega_h t) \cos(k_e x - \omega_e t)$

using identity
$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$k_h = \frac{k_1 + k_2}{2}$ $\quad k_e = \frac{k_1 - k_2}{2}$ $\quad \omega_h = \frac{\omega_1 + \omega_2}{2}$ $\quad \omega_e = \frac{\omega_1 - \omega_2}{2}$

↳ low freq beat envelope modulating high freq $\quad f_h = \frac{f_1 + f_2}{2}$ - avg

↳ beat period and frequency $\quad T_b = \frac{2\pi}{\omega_1 - \omega_2}$ $\quad f_b = \frac{\omega_1 - \omega_2}{2\pi} = f_1 - f_2$

so beat freq is difference in frequencies f_1 and f_2

- 3D wave equation $\frac{\partial^2 \psi}{\partial t^2} = v^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) \Rightarrow \frac{\partial^2 \psi}{\partial t^2} = v^2 \nabla^2 \psi$

- plane waves
  ↳ waves propagating in fixed direction with displacement ψ uniform over a plane perpendicular to direction of propagat

general plane wave $\tilde{\Psi}(\vec{r},t) = \tilde{A} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

↳ wave vector $\vec{k} = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}$ $|\vec{k}| = \frac{2\pi}{\lambda}$ direction of k = direction of propagation

$\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$

- spherical waves
  ↳ at large distance from source $\gg \lambda$ we can approximate as a plane wave
  ↳ A of spherical wave varies as $A \propto \frac{1}{\sqrt{r}}$

- energy of travelling waves

undisturbed:

length $l = x_0 + x_1$

speed $v = 0$

tension $T_0 = k_s x_1$

kinetic E $KE = 0$

potential E $PE = U_0 = \frac{1}{2} k_s x_1^2$

disturbed:

length $l + \delta l$

speed $v = \frac{\partial \psi}{\partial t}$

tension $T_0 = k_s x_1$ assume $\delta l \ll x_1$

kinetic E $KE = \frac{1}{2} \rho_0 l \left(\frac{\partial \psi}{\partial t}\right)^2$

potential E $PE = U_0 + \rho_0 v^2 l \left(\frac{\partial \psi}{\partial x}\right)^2$

remember $\sin\theta \approx \theta \approx \frac{\partial \psi}{\partial x} = -\frac{1}{v}\frac{\partial \psi}{\partial t}$

↳ kinetic energy per length $k = \frac{1}{2} \rho_0 \left(\frac{\partial \psi}{\partial t}\right)^2$

potential energy per length $u = \frac{1}{2} \rho_0 v^2 \left(\frac{\partial \psi}{\partial x}\right)^2$

↳ if $\psi = f(x - vt) \Rightarrow \frac{\partial \psi}{\partial x} = -\frac{1}{v}\frac{\partial \psi}{\partial t} \Rightarrow u = \frac{1}{2} \rho_0 \left(\frac{\partial \psi}{\partial t}\right)^2$

so for wave $\psi = f(x - vt)$ potential and kinetic energies are =

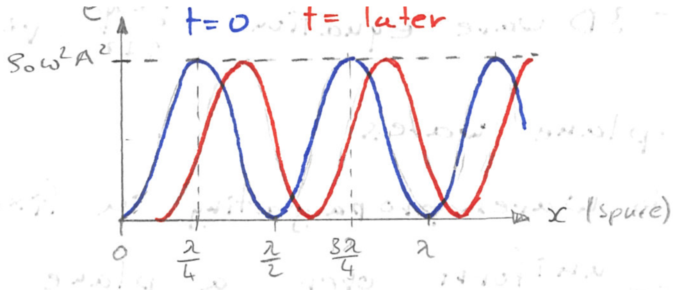
↳ total energy per unit length $e = u + k = \rho_0 \left(\frac{\partial \psi}{\partial t}\right)^2$

for sinosodial wave $\psi = A\cos(kx - \omega t)$ $e = \rho_0 \omega^2 A^2 \sin^2(kx - \omega t)$

↳ total energy per unit length

$$e = \rho_0 \omega^2 A^2 \sin^2(kx - \omega t)$$

so energy moves to the right along with the wave at same speed



- energy in one wavelength

$$E = \int_0^\lambda e(x)\,dx = \rho_0 \omega^2 A^2 \frac{\pi}{k} \implies E \propto A^2$$

$$v = \frac{\omega}{k}$$

- power of a wave

$$\langle P \rangle = \frac{E}{T} \quad T = \frac{\omega}{2\pi} \implies \langle P \rangle = \frac{1}{2} \rho_0 \omega^2 A^2 v$$

- wave impedance

force of driver $\vec{F}_{LR}$ left → right power $P = (\vec{F}_{LR})_y \frac{\partial \psi}{\partial t}$

by Newtons 3rd law $\vec{F}_{LR} = -\vec{F}_{RL}$

$$(\vec{F}_{RL})_y = T_0 \sin\theta = -\frac{T_0}{v} \frac{\partial \psi}{\partial t}$$

↳ define wave impedance $Z = \frac{T_0}{v} = \sqrt{\rho_0 T_0} = \rho_0 v$

which relates wave speed and force applied

↳ high wave impedance ⟹ lot of force has to be applied

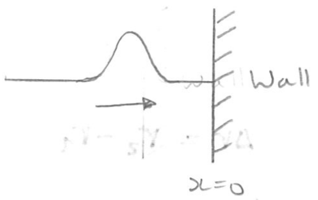
$P = Z \left(\frac{\partial \psi}{\partial t}\right)^2$ for a sinusodial wave $\langle P \rangle = \frac{1}{2} Z \omega^2 A^2$

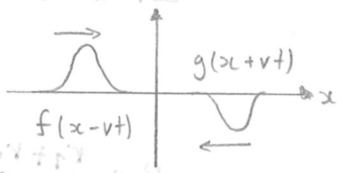
- properties of a wave v and z depend on

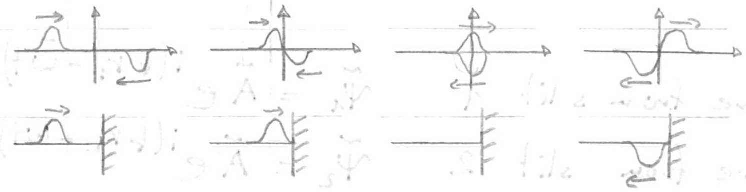
↳ T_0 - ellasticity which provides restoring force

↳ ρ_0 - inertia which opposes the restoring force

- reflection of travelling waves
  - ↳ string fixed at $x = 0$ so introduce constaint $\psi(x=0,t) = 0$ for all t
  - ↳ general solution $\psi(x,t) = f(x - vt) + g(x + vt)$ where $g(x+vt)$ is a second inverted wave travelling in $-x$

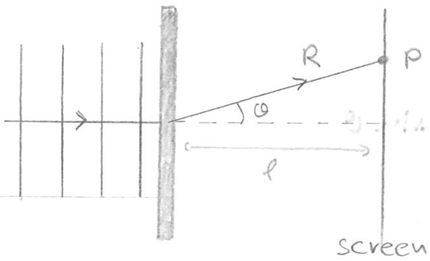






  - ↳ wave is allways reflected and inverted at boundary

- Diffraction – bending of waves around edges of an obstacle or an aperture
- Interference – superposition of waves to give resulting wave



wave arriving at P: $\tilde{\Psi}_{tot} = \tilde{A}_{tot}(\theta)\, e^{i(kR - \omega t)}$

R - distance of P from screen (slit)

$\tilde{A}$ - amplitude which varies with θ so position P

$k = \frac{\omega}{c}$

intensity at P $I = \frac{\langle Power \rangle}{area} \propto A^2 \implies I \propto \langle Re(\tilde{\Psi}_{tot})^2 \rangle$

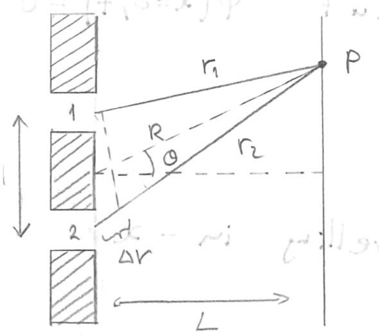
- time average theorem
  - ↳ if function $\tilde{P}$ has form $\tilde{P}(\vec{r},t) = \tilde{p}(\vec{r})\, e^{-i\omega t}$

    then $\langle Re(\tilde{P})^2 \rangle = \frac{1}{2} \tilde{p}\, \tilde{p}^*$
  - ↳ from this $\tilde{\Psi}_{tot} = \underbrace{\tilde{A}_{tot}\, e^{ikR}}\, e^{-i\omega t}$ → this factor averages out

$$I = \langle Re(\tilde{\Psi})^2 \rangle = \frac{1}{2} \tilde{A}_{tot}\, e^{ikR}\, \tilde{A}^*_{tot}\, e^{ikR} = \frac{1}{2} |\tilde{A}_{tot}|^2$$

$$I = \frac{1}{2} |\tilde{A}_{tot}|^2$$

- double slit interference



↳ assumptions
- $L \gg d$
- $\Delta r = d \sin\theta$
- r_1 and r_2 to be parallel
- amplitudes are same for r_1 and r_2 so same attenuation

wave from slit 1 $\quad \tilde{\Psi}_1 = \tilde{A} e^{i(kr_1 - \omega t)}$

wave from slit 2 $\quad \tilde{\Psi}_2 = \tilde{A} e^{i(kr_2 - \omega t)}$

$R = \frac{r_1 + r_2}{2} \qquad \Delta r = r_2 - r_1$

so $r_1 = R - \frac{\Delta r}{2}$

$r_2 = R + \frac{\Delta r}{2}$

$$\tilde{\Psi}_{tot} = \tilde{A}\left(e^{i(kr_1 - \omega t)} + e^{i(kr_2 - \omega t)}\right)$$

$$= \tilde{A}\left(e^{i(kR - \frac{k\Delta r}{2} - \omega t)} + e^{i(kR + \frac{k\Delta r}{2} - \omega t)}\right)$$

$$= \tilde{A}\, e^{i(kR - \omega t)}\, 2\cos\left(\frac{k\Delta r}{2}\right)$$

$$\tilde{\Psi}_{tot} = \tilde{A}_{tot}\, e^{i(kR - \omega t)} \qquad \tilde{A}_{tot} = 2\tilde{A}\cos\left(\frac{k\Delta r}{2}\right)$$

↳ intensity $I = \frac{1}{2}|\tilde{A}_{tot}|^2 \qquad \frac{k\Delta r}{2} = \underbrace{\frac{2\pi}{\lambda}}_{k} \frac{1}{2} \underbrace{d\sin\theta}_{\Delta r}$

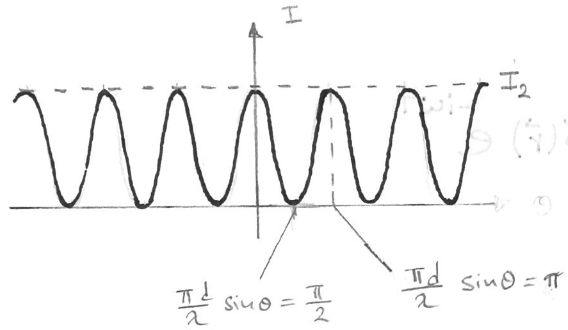
$$= 2|\tilde{A}|^2 \cos^2\left(\frac{k\Delta r}{2}\right)$$

$$I(\theta) = I_0 \cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right)$$

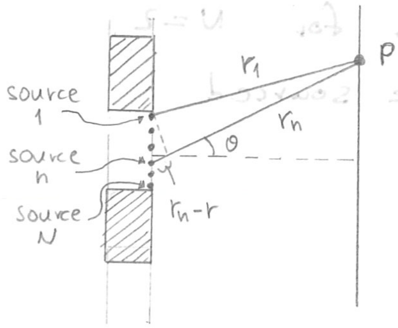
use SAA $\sin\theta = \theta$

1st min : $\frac{\pi d}{\lambda}\sin\theta = \frac{\pi}{2} \implies \theta = \frac{\lambda}{2d}$

1st max : $\frac{\pi d}{\lambda}\sin\theta = \pi \implies \theta = \frac{\lambda}{d}$



- single slit diffraction



D – slit width

N – total number of sources

d – separation of sources $\quad d = \frac{D}{N}$

path difference: $r_n - r_1 = (n-1) d \sin\theta$

$(n-1)\frac{D}{N}\sin\theta$

distance from centre of slit to P is R

$$R = \frac{1}{2}(r_1 + r_N) = r_1 + \frac{N-1}{2} d\sin\theta$$

wave from source n: $\quad \tilde{\Psi}_n = \tilde{A} e^{i(kr_n - \omega t)}$

$$\tilde{\Psi}_{tot} = \tilde{A}\left(e^{i(kr_1-\omega t)} + e^{i(kr_2-\omega t)} + e^{i(kr_3 - \omega t)} + \dots + e^{i(kr_N - \omega t)}\right)$$

$$= \tilde{A} e^{i(kr_1 - \omega t)}\left(1 + x + x^2 + \dots x^{N-1}\right) \quad x = e^{i\xi} \quad \xi = k d \sin\theta$$

from geometric series $\quad 1 + x + x^2 + \dots x^{N-1} = \frac{x^N - 1}{x-1}$

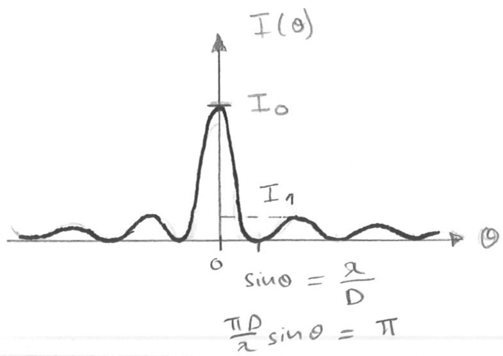
$$\tilde{\Psi}_{tot} = \tilde{A} e^{i(kr_1 - \omega t)}\left(\frac{e^{i\frac{N\xi}{2}}}{e^{i\frac{\xi}{2}}}\right)\left(\frac{e^{i\frac{N\xi}{2}} - e^{-i\frac{N\xi}{2}}}{e^{i\frac{\xi}{2}} - e^{-i\frac{\xi}{2}}}\right)$$

$$\tilde{\Psi}_{tot} = \tilde{A}_{tot}\, e^{i(kR - \omega t)} \quad \text{and} \quad \tilde{A}_{tot} = \tilde{A}\,\frac{\sin\left(\frac{\pi D}{\lambda}\sin\theta\right)}{\sin\left(\frac{\pi D}{N\lambda}\sin\theta\right)}$$

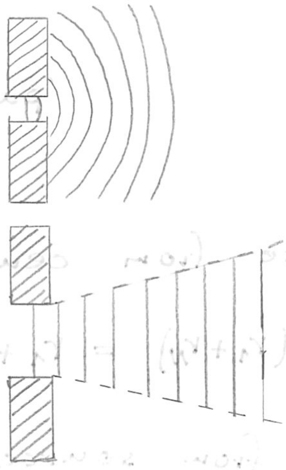
↳ intensity $\quad I(\theta) = \frac{1}{2}|\tilde{A}_{tot}|^2 = \frac{1}{2}|\tilde{A}|^2 \frac{\sin^2\left(\frac{\pi D}{\lambda}\sin\theta\right)}{\sin^2\left(\frac{\pi D}{N\lambda}\sin\theta\right)}$

↳ use SAA $\sin\theta \simeq \theta \qquad I_0 = \frac{1}{2}|\tilde{A}|^2 N^2$

↳ take limit $N \to \infty \qquad \underline{I(\theta) = I_0 \,\mathrm{sinc}^2\left(\frac{\pi D}{\lambda}\sin\theta\right)}$



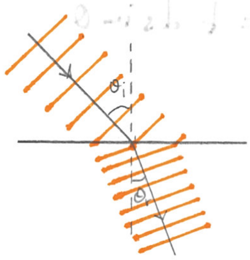
- for $N=2$ we get results as expected for double slit
- for $N=8$ we get twice as many minima as for $N=2$ since there are now more ways to combine the sources destructively



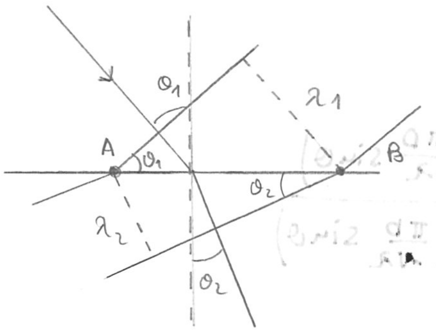
- for larger values of N more minima

for $D \gg \lambda$ waves dont bend since there is allways destructive interference away from central beam

- refraction
  ↳ change in direction of propagation of wave at interface between media of different wave speeds (ref indexes)

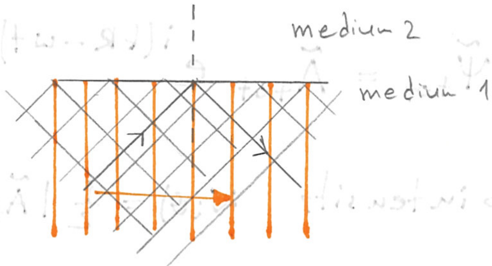


$$f_1 = \frac{v_1}{\lambda_1} \qquad f_2 = \frac{v_2}{\lambda_2} \qquad \frac{\lambda_1}{\lambda_2} = \frac{v_1}{v_2}$$



$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{\lambda_1}{\lambda_2} = \frac{v_1}{v_2} \qquad n_1 \sin\theta_1 = n_2 \sin\theta_2$$

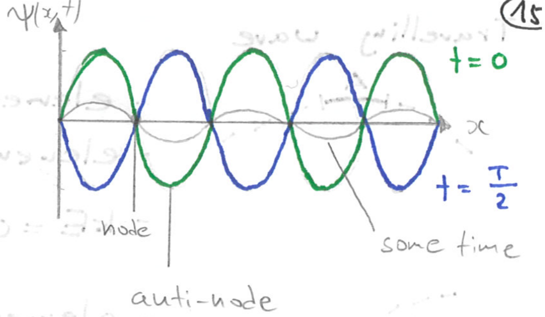
since $n = \frac{c}{v}$



- total internal reflection
  ↳ if $\theta_2 = \frac{\pi}{2}$ $\quad \theta_1 = \theta_{crit} \quad \sin(\theta_{crit}) = \frac{n_2}{n_1}$
  ↳ all energy in incident wave is reflected back
  ↳ incident + reflected waves interfere and resultant wave travells parallel to boundary

- standing waves

↳ superpose two counter propagating sinosodial travelling waves of equal magnitude



$$\Psi = \underbrace{\Psi_0 \cos(kx - \omega t + \phi_f)}_{\text{forwards}} + \underbrace{\Psi_0 \cos(kx + \omega t + \phi_b)}_{\text{backwards}}$$ ← (separable in x,t)

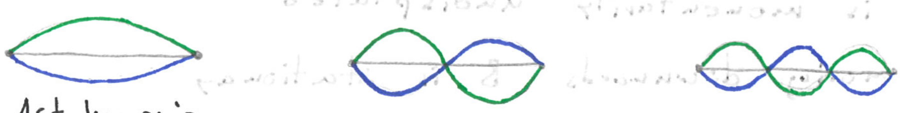
$$= 2\Psi_0 \cos(kx + \phi_1)\cos(\omega t + \phi_2) \qquad \phi_1 = \frac{\phi_f + \phi_b}{2} \qquad \phi_2 = \frac{\phi_f - \phi_b}{2}$$

↳ allways are periodic, all parts oscillate at same freq

↳ don't transport energy but rather store it

- stretched string

↳ length - ℓ mass - m $v = \frac{\omega}{k}$ $v = \sqrt{\frac{T}{\rho}}$ $\Rightarrow$ $\omega = k\sqrt{\frac{T}{\rho}}$



1st harmonic $\lambda_1 = 2L$ (fundemental)

2nd harmonic $\lambda_2 = L$

3rd harmonic $\lambda_3 = \frac{2}{3}L$

$$\lambda_n = \frac{2L}{n}$$

$$k_n = \frac{n\pi}{L}$$

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}} = n\omega_1$$

- energy of standing waves

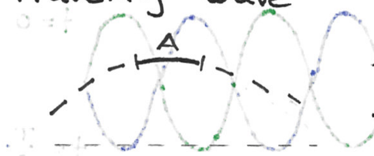
↳ general form of standing wave $\phi = 0$, $\Psi(x,t) = A\sin(kx)\cos(\omega t)$ (or swapped)

$$\frac{\partial \Psi}{\partial t} = -\omega A \sin(kx)\sin(\omega t) \Rightarrow \mathcal{K} = \frac{1}{2}\rho_0 \omega^2 A^2 \sin^2(kx)\sin^2(\omega t)$$

$$\frac{\partial \Psi}{\partial x} = kA\cos(kx)\cos(\omega t) \Rightarrow u = \frac{1}{2}\rho_0 v^2 k^2 A^2 \cos^2(kx)\cos^2(\omega t)$$

so generally $\mathcal{K} \neq u$ potential and kinetic energy per lentgth aren't equal

- Travelling wave



- element instantenously at rest
- element has undisplaced length

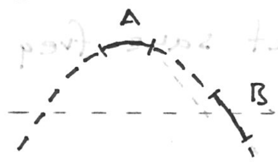
$kE = 0 \quad U = 0$

- element moving with max speed
- element at greatest angle so stretched

$kE = max \quad U = max$

- standing wave

$t = 0$

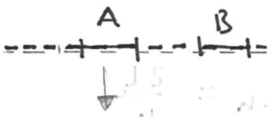


- speed of string everywhere momentarily 0
- A is not stretched but B is

A: $kE = 0 \quad U = 0$

B: $kE = 0 \quad U = max$

$t = \frac{T}{4}$



- string is momentarily undisplaced
- A is moving downwards B is stationary

A: $kE = max \quad U = 0$

B: $kE = 0 \quad U = 0$

- eigenvalue equations
  - ↳ to find normal modes (NM) of a system
  - ↳ at normal mode all parts oscillate with the same ω

$$\Psi(x,t) = \bar{\Psi}(x) e^{-i\omega t} \Rightarrow \frac{\partial^2 \Psi}{\partial t^2} = -\omega^2 \bar{\Psi}(x) e^{-i\omega t} \qquad \frac{\partial^2 \Psi}{\partial x^2} = e^{-i\omega t} \frac{d^2 \bar{\Psi}}{dx^2}$$

substituting back into wave equation

$$\frac{\partial^2 \Psi}{\partial t^2} = v^2 \frac{\partial^2 \Psi}{\partial x^2} \qquad v = \sqrt{\frac{T}{\rho_0}}$$

$$-\omega^2 \bar{\Psi}(x) e^{-i\omega t} = \frac{T}{\rho} e^{-i\omega T} \frac{d^2 \Psi}{dx^2} \Rightarrow -\frac{T}{\rho} \frac{d^2 \bar{\Psi}}{dx^2} = \omega^2 \bar{\Psi}$$

↳ gives eigen value equation of general form

$F(\bar{\Psi}) = \Lambda \bar{\Psi}$ — F - differential operator

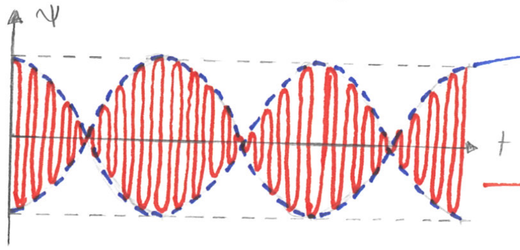
Λ - eigenvalue

$\bar{\Psi}$ - eigen function

- solving stretched string eigenvalue problem
  - ↳ boundary conditions $\Psi(x=0)=0$ $\Psi(x=L)=0$
  - ↳ trial solution $\Psi(x) = A\sin(kx)$ $\frac{d^2\Psi}{dx^2} = -k^2A\sin(kx)$
  - ↳ substitute into $-\frac{T}{\rho}\frac{d^2\Psi}{dx^2} = \omega^2\Psi$

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}} = k_n\sqrt{\frac{T}{\rho}} \qquad k_n = \frac{n\pi}{L}$$

- phase and group velocity



peaks on envelope travel at a group velocity $V_g = \frac{d\omega}{dk}$

peaks on modulated wave travel at phase velocity $V_p = \frac{\omega}{k}$

↳ when wave carries information they are carried within the envelope so travel at group velocity

↳ phase velocity depends on medium $V_p = \sqrt{\frac{T}{\rho}}$

↳ if V_p is independant of ω $V_p = V_g$

- dispersive medium
  - ↳ medium in which V_p varies with ω $V_p = V_p(\omega)$ so $V_g \neq V_p$
  - ↳ for dispersive medium ω and k are linked by dispersion relation. for non-dispersive $\omega \propto k$

e.g.) for plasma $\omega = \sqrt{\omega_p^2 + k^2c^2}$ ω_p - plasma frequency

so in plasma waves oscillate at $V_p > c$! but information still travels at $V_g < c$

↳ glass is dispersive so V_p decreases with ω

$n = \frac{c}{V_p}$ so $n\uparrow$ as $\omega\uparrow$

dispersion in glass :