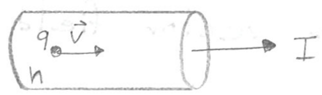
Magnetism 2022

- current density
  - consider material with different particle species 1, 2, 3, N

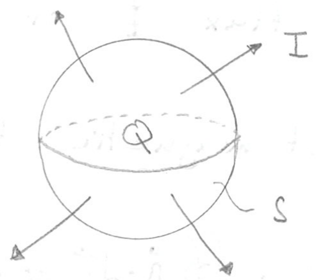
    n_i - number density q_i - charge $\vec{v}_i$ - drift velocity
  - species could be e^- in wire, +ve holes in semi conductor ions in electrolyte, p^+ and e^- in plasma
  - assume no variation in speed due to temp

  current density: $\vec{j} = \sum_{i=1}^{N} n_i q_i \vec{v}_i$ or just $\vec{j} = nq\vec{v}$

- electric current
  - charge passing through surface per unit time

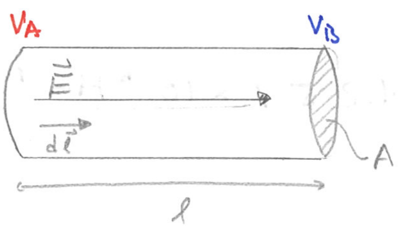


$$I = \iint_S \vec{j} \cdot d\vec{s}$$



- conservation of charge $\oiint_S \vec{j} \cdot d\vec{s} = -\frac{dQ}{dt}$
  - by using divergence theorem and writing Q in terms of charge density ρ $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$

- Ohm's law



$V = V_A - V_B$ resistivity - η

conductivity - $\sigma = \frac{1}{\eta}$

- Ohm's law: $\vec{E} = \eta \vec{j}$

$$\int_A^B \vec{E} \cdot d\vec{l} = \int_A^B \eta \vec{j} \cdot d\vec{l}$$

since $\vec{j}$ is constant along l

$$V = \eta j l$$

$$= \eta \frac{I}{A} l$$

define $R = \frac{\eta l}{A}$

$$V = IR$$

- Joule heating
  - e^- are accelerated by $\vec{E}$ between collisions and overall move with average constant speed
  - force applied by $\vec{E}$ does work on the e^- at rate $\vec{F}\cdot\vec{v}$
  - heating rate per unit volume

$$p = \vec{F}\cdot\vec{v} = \eta(-e\vec{E})\vec{v} = \vec{j}\cdot\vec{E} = \eta(\vec{j})^2$$

for conductor length l area A

$$P = \eta j^2 A l = \eta\left(\frac{I}{A}\right)^2 A l = \frac{\eta l}{A} I^2$$

$$P = I^2 R$$

- Magnetic field
  - vector field $\vec{B}$ defined over all space units Tesla $T = kg\,s^{-2}A^{-1}$

- Magnetic flux
  - flux Φ of $\vec{B}$ through surface S $\Phi = \iint_S \vec{B}\cdot d\vec{s}$ units Wesser $W = Tm^2$

- net magnetic flux through closed surface $= 0$

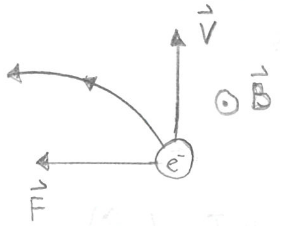
$$\Phi = \oiint_S \vec{B}\cdot d\vec{s} = 0 \xrightarrow[\text{theorem}]{\text{divergence}} \nabla\cdot\vec{B} = 0$$

  - magnetic fields aren't generated by stationary charge no monopoles

- permiasility of free space $\mu_0 = \frac{2\alpha}{e^2}\frac{h}{c} = 1.257\times10^{-6}\,Hm^{-1}$
  - use approximation $\mu_0 \approx 4\pi\times10^{-7}\,Hm^{-1}$

- Lorentz force $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$
  - non-conservative since $\vec{F} = \vec{F}(\vec{v})$ if $\vec{E} = 0$ $\vec{F}\perp\vec{v}$
  - magnetic field does no work
  - field lines of $\vec{B}$ are not lines of magnetic force

- particle motion in magnetic field
  - ↳ consider a charged particle moving $\perp$ to uniform $\vec{B}$

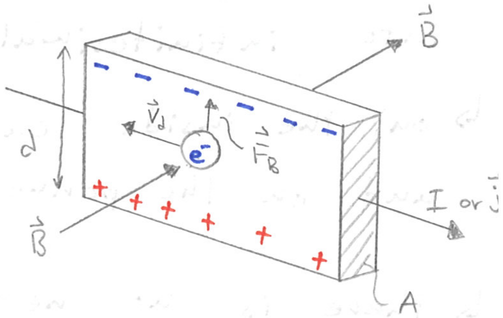


e^- so q is -ve

force is allways $\perp$ to $\vec{v}$ and $\vec{B}$ and constant so gives circular motion

  - ↳ cyclotron frequency $\omega = \dfrac{qB}{m}$ independant of v
  - ↳ cyclotron / larmor radius $r_L = \dfrac{mv}{qB}$

- Hall effect



  - ↳ charges flow so $\vec{j} = nq\vec{v}$ $\quad I = nqvA$
  - ↳ Lorentz force deflects charges $\perp$ to $\vec{v}$ so $\vec{j}$ and $\vec{B}$ inducing a voltage across the conductor and an electric field $\vec{E}$ opposite to $\vec{F_B}$
  - ↳ equilibrium forms when $\vec{F} = 0$

$$\vec{F} = 0 = q(\vec{E} + \vec{v} \times \vec{B}) \implies \vec{E} = -\vec{v} \times \vec{B} \qquad E = \frac{IB}{Anq} \qquad V = \frac{IBd}{Anq}$$

- current in a thin wire
  - ↳ consider n charges per unit length

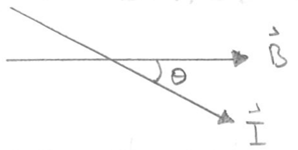
$$I = nvq \qquad n = \frac{Q}{\ell} \qquad \vec{v} = v\,d\hat{\ell}$$

- Forces on a wire
  - ↳ for length $d\ell$ along wire charge is $qnd\ell$ with velocity $\vec{v} = v\,d\hat{\ell}$

$$\vec{F} = q\,\vec{v} \times \vec{B} = qn\,d\ell \; v\,d\hat{\ell} \times \vec{B} = qnv\; d\vec{\ell} \times \vec{B}$$

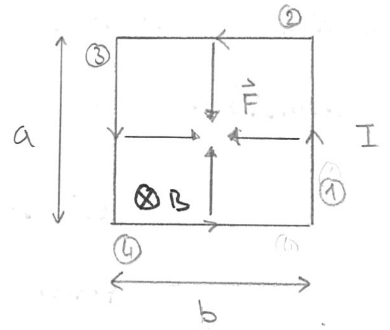
$$\vec{F} = I\int_0^L d\vec{\ell} \times \vec{B} = -I\int_0^L \vec{B} \times d\vec{\ell}$$

  - ↳ for a straight wire in a uniform field



$$\vec{F} = I\,\vec{L} \times \vec{B} = IBL\sin\theta$$

- current loops
  - in a real circuit current must flow in a closed loop
  - force acts to collapse the loop in on itself
  - for a rigid rectangle there is no net force

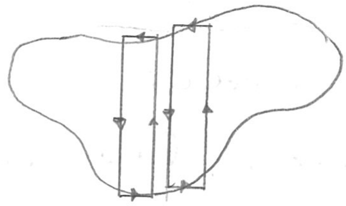


$\vec{F}_1 = BIa(-\hat{x})$

$\vec{F}_3 = BIa(\hat{x})$

so cancel out

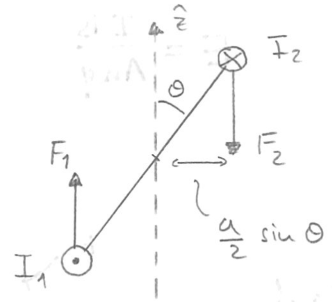
- arbitrary current loops
  - any complex shape can be decomposed into infinitesimal rectangles
  - on the inside currents cancel out and on the perimeter they add up



  - there is no net force on any flat closed current loop in a uniform magnetic field

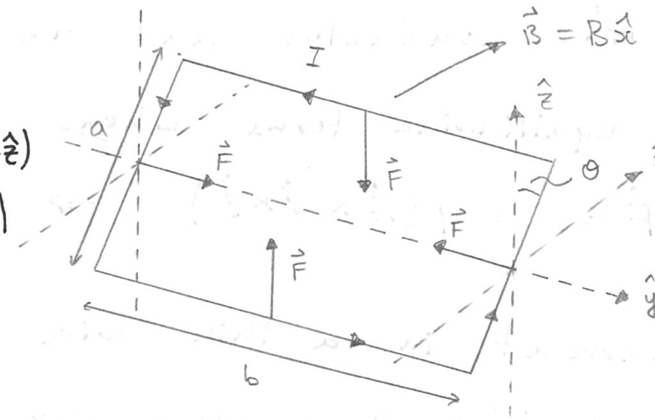
- Torque on a current loop

view along $\hat{y}$:



$\vec{F}_2 = BIb(-\hat{z})$

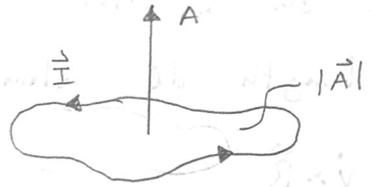
$\vec{F}_1 = BIb(\hat{z})$



  - torque $\Gamma = F d_\perp$

$\Gamma = BIA\sin\theta \qquad A = ab$

$$\Gamma = BIb\frac{a}{2}\sin\theta + BIb\frac{a}{2}\sin\theta$$
$$= BIab\sin\theta$$

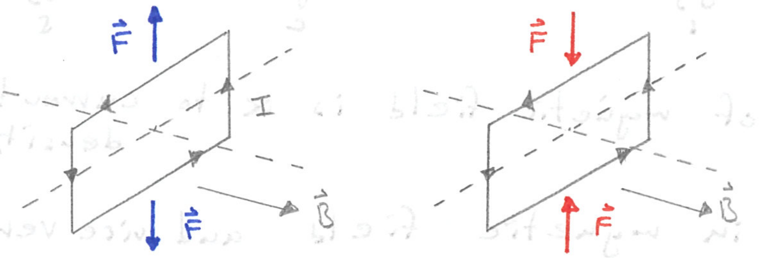


- Magnetic dipole moment
  - define vector area $\vec{A}$ using RHR with respect to $\vec{I}$
  - define magnetic dipole moment $\vec{\mu} = I\vec{A}$

$$\vec{\Gamma} = \vec{\mu} \times \vec{B}$$

  - due to movement e^- exhibit a dipole moment in some elements this can be used to apply a torque on the e^- using a magnetic field

- work done by torque on a loop
  - ↳ when loop is vertical $\vec{A} \parallel \vec{B}$ torque is 0 so eqb



stable equilibrium — unstable equilibrium

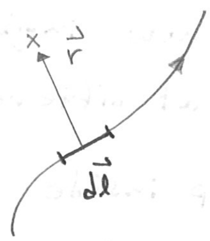
- rotating loop from stable equilibrium we do work on the loop
- rotating loop from unstable equilibria loop does work

  - ↳ work done is conservative $W = W(\theta)$

$$W = \int_0^\theta |W|\, d\theta = \int_0^\theta \mu B \sin\theta \, d\theta \qquad W = \mu B (1-\cos\theta)$$

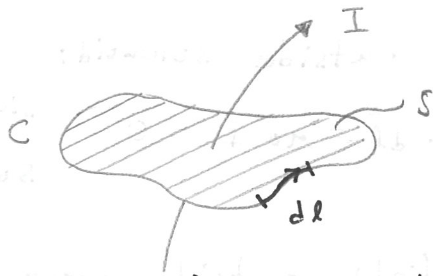
- electric motor
  - ↳ use slip ring commutator to reverse current so always unsta
  - ↳ to increase power : - stronger field $\uparrow B$
    - more turns of wire $\uparrow \mu$
    - more loops at different angles $\uparrow \mu$
  - ↳ unlike internal combustion engine spinning faster doesn't ↑ power

- Biot - Savart law
  - ↳ magnetic field element $d\vec{B}$ at a distance $\vec{r}$ from a current element $I\, d\vec{l}$



$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2} \implies \vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \hat{r}}{r^2}$$

- Ampére's law integral form
  - ↳ same principle as Biot - Savart
  - ↳ current I passing through a loop C



$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I \implies \oint_C \vec{B} \cdot d\vec{l} = \mu_0 \iint_S \vec{j} \cdot d\vec{s}$$

$d\vec{l}$ and I related by RHR

  - ↳ only valid in absence of time varying electric fields

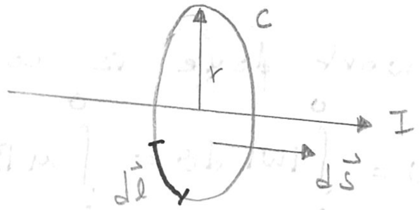
- Ampére's law differential form

apply stoke's theorem $\oint_c \vec{P}\cdot d\vec{l} = \iint_s \nabla\times\vec{P}\cdot d\vec{s}$ to $\oint_c \vec{B}\cdot d\vec{l} = \mu_0 \iint_s \vec{J}\cdot d\vec{s}$

↳ $\nabla \times \vec{B} = \mu_0 \vec{J}$ so curl of magnetic field is $\propto$ to current density

↳ current generates "twist" in magnetic field and vice versa

- Ampére's law for ∞ straight wire



↳ due to symmetry only consider ϕ component

$\oint_c \vec{B}\cdot d\vec{l} = \mu_0 I \Rightarrow 2\pi r B_\phi = \mu_0 I$

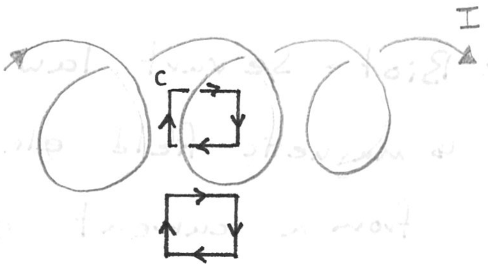
$$B_\phi = \frac{\mu_0 I}{2\pi r}$$

↳ B_θ component is zero since $\nabla\cdot\vec{B} = 0$

↳ component of B along wire is zero by symmetry

- Ampére's law for a solenoid

↳ consider a long solenoid with N turns per meter carrying a current I



↳ now imagine an Amperian wire loop at different positions around the solenoid

loop inside solenoid:

$\oint_c \vec{B}\cdot d\vec{l} = \mu_0 I = 0$ since current through loop is 0 so $\vec{B}$ inside must be constant

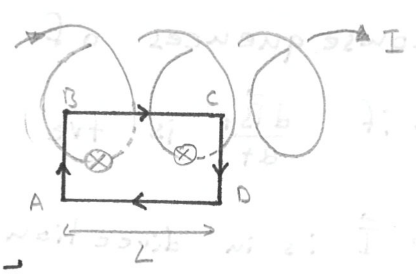
loop outside solenoid:

$\oint_c \vec{B}\cdot d\vec{l} = \mu_0 I = 0$ since current through loop is 0 so $\vec{B}$ outside must be constant

if field outside were non-zero it would have ∞ energy

so $\vec{B}$ outside must be zero

loop crossing solenoid:



total current through loop $I_{loop} = ILN$

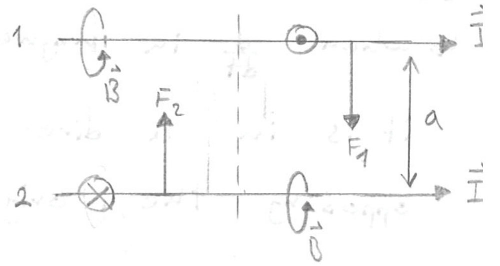
$\int_{AB} \vec{B}\cdot d\vec{l} = 0$ and $\int_{CD} \vec{B}\cdot d\vec{l} = 0$ since they are not along solenoid and $\nabla \times \vec{B} = 0$

$\int_{AD} \vec{B}\cdot d\vec{l} = 0$ since its outside $\int_{BC} \vec{B}\cdot d\vec{l} = B_x L$ so $\oint_C \vec{B}\cdot d\vec{l} = B_x L$

$B_x L = ILN \Rightarrow B = \mu_0 NI$ along solenoid

↳ for ∞ solenoid B inside $B = \mu_0 NI$ and is 0 outside

- Forces between 2 current carrying wires



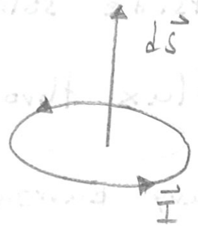
field due to wire 1 $B_1 = \dfrac{\mu_0 I}{2\pi a}$

wire 2 feels the field over length L $F_2 = B_1 I L$

↳ force per unit length on wire 2 $f_2 = \dfrac{\mu_0 I}{4\pi a}$

↳ current flowing in same direction wires attract

- Faraday's law



↳ emf induced in a loop is equal to the rate of change of magnetic flux through the loop

$$\mathcal{E} = -\frac{d\Phi}{dt} \xrightarrow[\Phi = \iint \vec{B}\cdot d\vec{s}]{\mathcal{E} = \oint \vec{E}\cdot d\vec{l}} \oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{s}$$

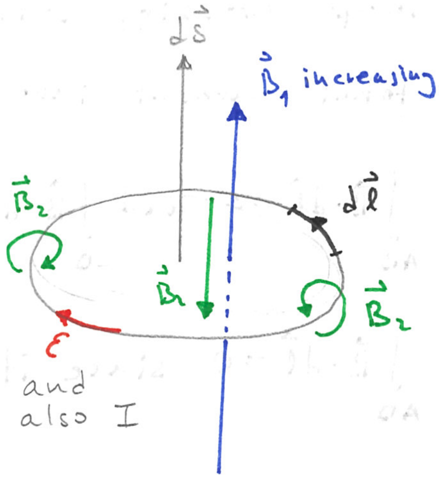
↳ apply stoke's theorem to get differential form $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

↳ changing magnetic field in time will induce circulation in the electric field

↳ if there are no mobile charges (no wire) and changing $\vec{B}$ an $\mathcal{E}$ is still induced around some closed path but since there arent mobile charges there isnt a way to check $\mathcal{E}$

- consequences of Faraday's law

↳ if $\frac{d\vec{B}_1}{dt}$ is +ve $\mathcal{E}$ is -ve by RHR



↳ $\vec{I}$ is in direction of $\mathcal{E}$ (which is -ve) and results in a secondary field $\vec{B}_2$ around the wire

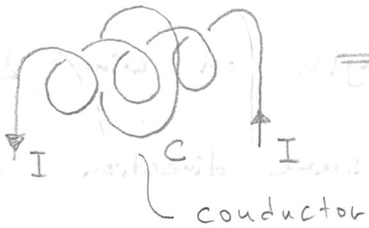
↳ $\vec{B}_2$ inside the loop opposes the increasing primary magnetic field $\vec{B}_1$

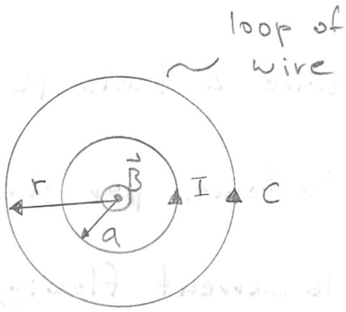
↳ this conserves energy preventing a runaway

- Lenz's law

↳ when $\frac{d}{dt}$ in magnetic field produces an induced current it is in a direction as to produce a magnetic field opposing the original change

- induced electric fields





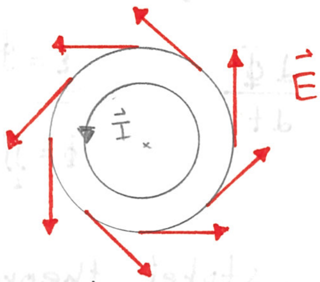
inside solenoid $B = \mu_0 I N$

outside solenoid $B = 0$

↳ flux through conductor C $\Phi = B\pi a^2 = \mu_0 I N \pi a^2$

↳ vary current $\frac{d\Phi}{dt} = \mu_0 N \pi a^2 \frac{dI}{dt}$ and apply Faradays law

↳ $E_\phi = -\frac{\mu_0 N}{2} \frac{a^2}{r} \frac{dI}{dt}$



- properties of induced electric fields

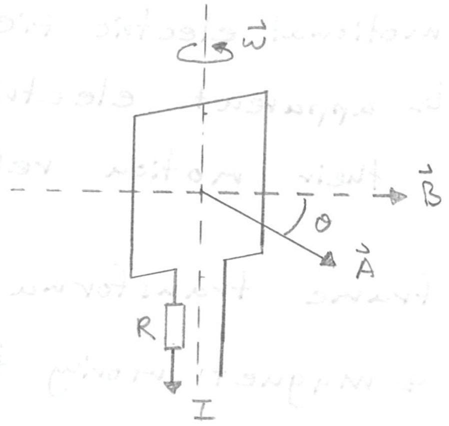
↳ different to $\vec{E}$ from static charges

↳ are circulating $\nabla \cdot \vec{E} = 0$ so have no start or end

↳ non-conservative, gain energy by going around them

↳ electric fields can exist even in locations where $\vec{B} = 0$ so $\frac{d\vec{B}}{dt} = 0$ all that matters is that $\frac{d\Phi}{dt} \neq 0$ (through the loop in that region)

- generators
  - consider a loop area A with N turns rotating at ω



flux through all N loops : $\Phi = BAN\cos(\omega t)$

$$\mathcal{E} = -\frac{d\Phi}{dt} = BAN\omega\sin(\omega t) \qquad I = \frac{BAN\omega\sin(\omega t)}{R}$$

$$P = \frac{(NBA\omega)^2}{R}\sin^2(\omega t)$$

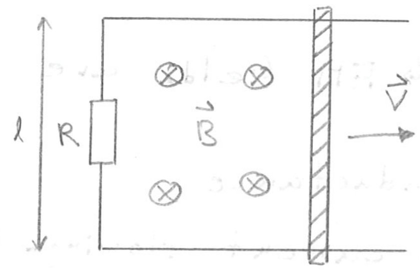
- work done on generator
  - current flows through loop so magnetic moment

$$\mu = NIA = \frac{N^2BA^2\omega}{R}\sin(\omega t) \qquad \text{torque: } \Gamma = \vec{\mu}\times\vec{B} = \mu B\sin(\omega t)$$

$$P = \Gamma\omega = \frac{(NBA\omega)^2}{R}\sin^2(\omega t)$$

- so power due to torque is equal to power output of genera

- slide wire generator

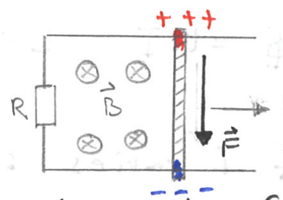


  - changing loop area $\frac{dA}{dt} = lv$

$$\mathcal{E} = -\frac{d\Phi}{dt} = B\frac{dA}{dt} = Bvl \qquad I = \frac{Bvl}{R}$$

  - dissipated power $P = \frac{(Bvl)^2}{R}$
  - power required to push rod $\vec{F} = I\vec{l}\times\vec{B}$ $\quad P = \vec{F}\cdot\vec{v} = \frac{(Bvl)^2}{R}$

- Motional electric field

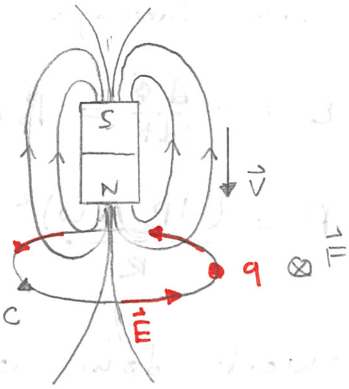


  - now think of the charges in the wire
  - charges move with wire so $\vec{v}$ and experience Lorentz force
  - charge separation causes motional electric field $\vec{E}$ which is in equilibrium with $\vec{B}$ $\quad \vec{E} = -\vec{v}\times\vec{B}$

potential along rod $\mathcal{E} = El = Bvl$ $\quad$ current $I = \frac{Bvl}{R}$

- motional electric field
  - ↳ apparent electric field which charges experience due to their motion relative to the magnetic field

frame transformations



↳ magnet moving towards loop C with $\vec{v}$

flux Φ ↑ so by Faradays law there is an electric field around C causing charges in the loop to experience a force.

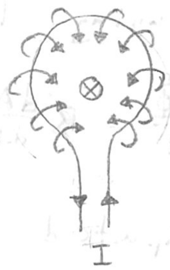
↳ loop moving towards stationary magnet

charges in loop are moving in magnetic field so experience a Lorentz force

↳ both frames are inertial so should have same physics but the forces seen to have a different cause

↳ EM fields are not invariant between inertial frames

- inductance



↳ current flowing through a loop generates flux Φ threading through the loop

↳ the flux is proportional to current   inductance is constant of pr

$\Phi = LI \qquad L = \frac{\Phi}{I} \qquad \text{Henry} = \text{Wb A}^{-1}$

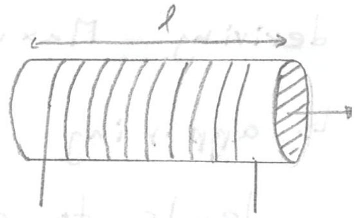
↳ if I varies so does Φ $\qquad \mathcal{E} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$

and induced $\mathcal{E}$ opposes change in I

↳ results in <u>inertia</u> in electric circuits

$L \propto A$ of loop so dont use large loops

- inductance of a solenoid



↳ long thin solenoid length l area A with N total turns

inside solenoid $B = \frac{\mu_0 N I}{l}$ $\qquad \Phi = NAB = \frac{\mu_0 N^2 A I}{l}$

$$L = \frac{\Phi}{I} = \frac{\mu_0 N^2 A}{l}$$

- energy density of magnetic field

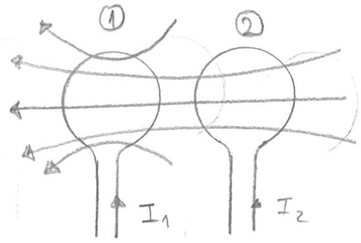
↳ an inductor carrying current stores energy in $\vec{B}$ field

$$U = \frac{1}{2} L I^2 = \frac{1}{2} \frac{\mu_0 N^2 A I^2}{l}$$

$$B = \frac{\mu_0 N I}{l} \implies I = \frac{B l}{\mu_0 N} \implies U = \frac{1}{2} \frac{1}{\mu_0} B^2 \underbrace{A l}_{\text{vol}}$$

↳ energy density of magnetic field $u = \frac{1}{2} \frac{1}{\mu_0} B^2$

- Mutual inductance



↳ consider two loops near eachother

current in loop 1 produces flux linking with loop 2

I_1 in loop 1 generates Φ_2 in loop 2 and vice versa

$$\Phi_2 = M_{21} I_1 \qquad \Phi_1 = M_{12} I_2 \qquad M_{21} = M_{12} = M$$

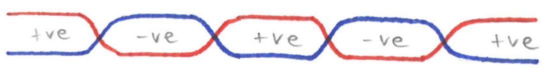
- Mutual induction

↳ changing current in loop 1 induces emf in loop 2

$$\mathcal{E}_2 = -\frac{d\Phi_1}{dt} = -M \frac{dI_1}{dt}$$

↳ M represents coupling between the two loops

- twisted pairs

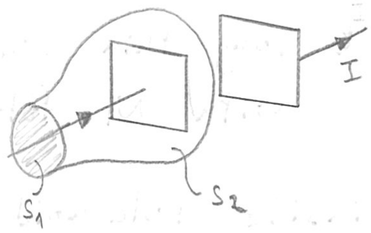


↳ area is constantly flipped so emf allways induced in opposite direction and cancels out

· deriving Maxwell - Ampére equation

↳ applying Ampéres law $\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \iint_S \vec{j} \cdot d\vec{s}$

leads to a paradox



for S_1: $\oint_C \vec{B} \cdot d\hat{l} = \mu_0 I = \mu_0 j S_1$ for S_2: $\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \iint_S \vec{j} \cdot d\vec{s} = 0$

↳ for a charging capacitor $q = CV$ $C = \frac{\varepsilon_0 A}{d}$ $\Rightarrow$ $q = \varepsilon_0 A E$

$\frac{V}{d} = E$

$\frac{dq}{dt} = \varepsilon_0 A \frac{dE}{dt}$ so $I = \varepsilon_0 \iint_S \frac{d\vec{E}}{dt} \cdot d\vec{s}$

this current is called a displacement current

↳ add to Ampéres law $\oint_C \vec{B} \cdot d\hat{l} = \mu_0 \left(\iint_S \vec{j} \cdot d\vec{s} + \varepsilon_0 \iint_S \frac{d\vec{E}}{dt} \cdot d\vec{s} \right)$

↳ differential form $\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$ $c^2 = \frac{1}{\mu_0 \varepsilon_0}$