Vector Operators

Gradient of a scalar field

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\text{grad } U = \nabla U = \frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}$$

- gradient of a scalar field is a vector field which tends to the point in direction of greatest change of field

e.g.) $U = r^2 \quad r^2 = x^2+y^2+z^2$

$$\nabla U = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k} \qquad \text{so} \quad \nabla U = 2\vec{r}$$

$U = \vec{c}\cdot\vec{r}$ where $\vec{c}$ is constant

$$\nabla U = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(c_1x + c_2y + c_3z)$$

$$= c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

$$= \vec{c} \qquad \text{so} \quad \nabla U = \vec{c}$$

$U = U(r) \quad r = \sqrt{x^2+y^2+z^2}$

$$\nabla U = \frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k} \qquad \frac{\partial U}{\partial x} = \frac{dU}{dr}\frac{\partial r}{\partial x},\quad \frac{\partial U}{\partial y} = \frac{dU}{dr}\frac{\partial r}{\partial x},\quad \frac{\partial U}{\partial z} = \frac{dU}{dr}\frac{\partial r}{\partial z}$$

$$= \frac{dU}{dr}\left(\frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}\right) \qquad \frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{\frac{1}{2}}$$

$$= \frac{1}{2}(2x)(x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$= \frac{dU}{dr}\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}\right) \qquad = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$= \frac{dU}{dr}\left(\frac{\vec{r}}{r}\right)$$

so $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

- if our position is $\vec{r}$ and we move infinitesimal distance $d\vec{r}$ the change in U is $dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz$

but $d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$ and $\nabla U = \hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}$

so change in U $\quad dU = \nabla U\cdot d\vec{r}$ $\quad$ now divide by ds $\quad \frac{dU}{ds} = \nabla U\cdot\frac{d\vec{r}}{ds}$

↳ ∇U has the property that rate of change of U wrt distance in a direction is the projection of ∇U onto that direction

- $\frac{dU}{ds}$ - directional derivative has different value for each direction so a direction has to be specified

at any point P, ∇U points in direction of greatest change of U at P and has magnitude equal to rate of change of U wrt distance in that direction

- for a surface of constant U $U(x,y,z) = \text{constant}$ if we move within the surface there is no change so $\frac{dU}{ds} = 0$ therefore for any $\frac{d\vec{r}}{ds}$

$\nabla U \cdot \frac{d\vec{r}}{ds} = 0$ so ∇U is allways normal to a surface of constant U

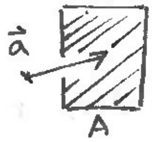
Divergence of a Vector field

$$\text{div}\,\vec{a} = \nabla \cdot \vec{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

since $\nabla \cdot \vec{a} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)$

$$= \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

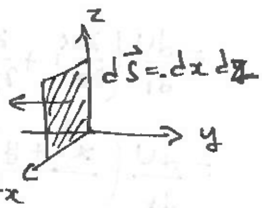
- consider a vector field such as fluid flow denoted $\vec{a}(\vec{r})$. The vector has magnitude equal to mass of water crossing a unit area perpendicular to the direction of $\vec{a}$ per unit time



$|\vec{a}| = \frac{m}{At}$

now consider infinitesimal volume element $dV = dx\,dy\,dz$ in cartesian co-ordinates and think of area $dx\,dz$ perpendicular to y axis facing outwards in $-$ve directn

surface area $d\vec{S} = -dx\,dz\,\hat{j}$. component of vector $\vec{a}$ normal to $d\vec{S}$ is $\vec{a}\cdot\hat{j} = a_y$ pointing inwards



contribution to outward flux is $\vec{a}\cdot d\vec{S} = -a_y(x,y,z)\,dz\,dx$

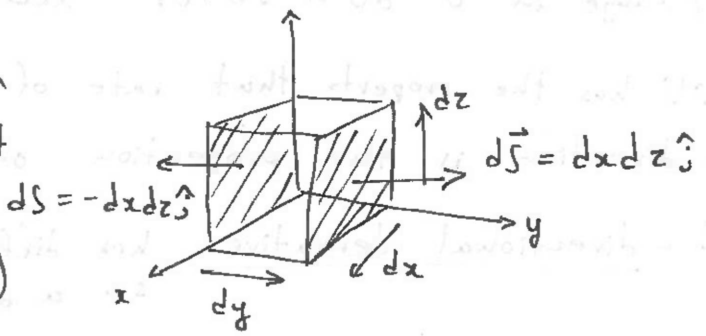
for the other side we have moved by amount dy so outward flux is $a_y(x, y+dy, z)\,dz\,dx = \left(a_y + \frac{\partial a_y}{\partial y}dy\right)dx\,dz$

total outward flux from these faces $\frac{\partial a_y}{\partial y}dy\,dx\,dz = \frac{\partial a_y}{\partial y}dV$

outward flux from all faces $= \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right)dV = \nabla\cdot\vec{a}\,dV$

↳ divergence of a vector field represents the flux generation per unit volume at each point of the field

(divergence is efflux not influx)



total efflux from infinitesimal vol = flux integrated over surface of the volume

Laplacian of a scalar field

$$\operatorname{div}(\operatorname{grad} U) = \nabla\cdot(\nabla U) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \quad \text{or just} \quad \nabla^2 U$$

(for $\nabla^2 \vec{a}$ gives a vector)

- the Laplace's equation $\nabla^2 U = 0$

$$\nabla^2 \frac{1}{r} = 0$$

e.g.) $U = \frac{1}{r} = (x^2+y^2+z^2)^{-\frac{1}{2}}$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{\partial}{\partial x} - x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$= -(x^2+y^2+z^2)^{-\frac{3}{2}} + 3x^2(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$= \frac{1}{r^3}\left(\frac{3x^2}{r^2} - 1\right)$$

adding up all the terms for x, y, z $\quad \nabla^2\frac{1}{r} = \frac{1}{r^3}\left(-3 + 3\frac{(x^2+y^2+z^2)}{r^2}\right) = 0$

Curl of a vector field

$$\operatorname{curl}(\vec{a}) = \nabla\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{i} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\hat{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{k}$$

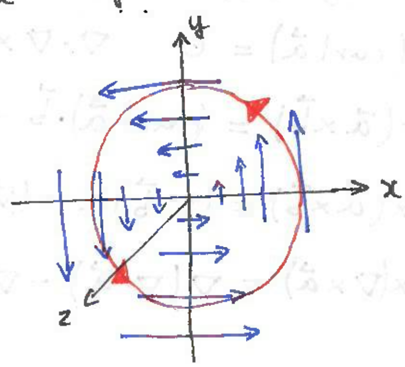
e.g.) $\vec{a} = -y\hat{i} + x\hat{j}$

$$\nabla\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}x\right)\hat{i} + \left(\frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x}0\right)\hat{j} + \left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y)\right)\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + (1+1)\hat{k}$$

$$= 2\hat{k}$$

$\vec{a} = -y\hat{i} + x\hat{j}$ sketched:



this gives a field with a curl $\nabla\times\vec{a} = 2\hat{k}$ in r-h screw out of the page

field like this must have a finite value to the line integral around a complete loop $\oint_C \vec{a}\cdot d\vec{r}$

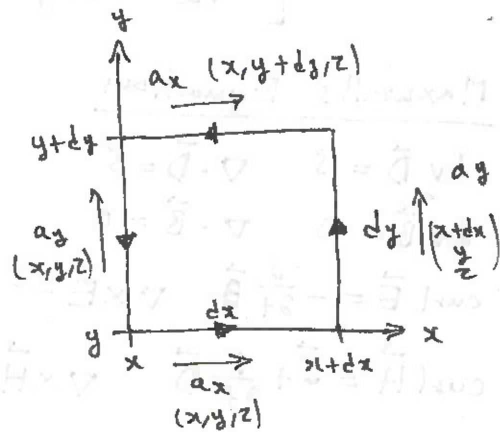
- circulation of a vector $\vec{a}$ round any closed course C is $\oint_C \vec{a}\cdot d\vec{r}$ and the curl of vector field $\vec{a}$ represents the vorticity

vorticity = circulation per unit area

proof:

$a_x(x,y,z)$ $\quad a_x(x, y+dy, z) = a_x(x,y,z) + \frac{\partial a_x}{\partial y}dy$

$a_y(x,y,z)$ $\quad a_y(x+dx, y, z) = a_y(x,y,z) + \frac{\partial a_y}{\partial x}dx$



starting at the bottom going anti-clockwise 4 contributions to circulation dC are:

$$dC = +(a_x dx) + (a_y(x+dx,y,z)dy) - (a_x(x,y+dy,z)dx) - (a_y dy)$$

$$= (a_x dx) + \left(a_y + \frac{\partial a_y}{\partial x}dx\right)dy - \left(a_x + \frac{\partial a_x}{\partial y}dy\right)dx - (a_y dy)$$

$$= \cancel{(a_x dx)} + \cancel{(a_y dy)} + \frac{\partial a_y}{\partial x}dx\,dy - \cancel{(a_x dx)} - \frac{\partial a_x}{\partial y}dx\,dy - \cancel{(a_y dy)}$$

$$= \frac{\partial a_y}{\partial x}dx\,dy - \frac{\partial a_x}{\partial y}dx\,dy$$

$$= \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)dx\,dy = (\nabla\times\vec{a})\cdot d\vec{S} \quad \text{where} \quad d\vec{S} = dx\,dy\,\hat{k}$$

Definitions

- vector field with zero divergence – solenoidal $\nabla\cdot\vec{a} = 0$
- vector field with zero curl – irrotational $\nabla\times\vec{a} = 0$
- scalar field with zero gradient – constant $\nabla U = 0$

Vector operator Identities

- $\text{curl}(\text{grad}\, U) = \vec{0}$ $\quad \nabla\times\nabla U = \vec{0}$
- $\text{div}(\text{curl}\,\vec{a}) = 0$ $\quad \nabla\cdot\nabla\times\vec{a} = 0$
- $\text{div}(\vec{a}\times\vec{b}) = (\text{curl}\,\vec{a})\cdot\vec{b} - \vec{a}(\text{curl}\,\vec{b})$ $\quad \nabla\cdot(\vec{a}\times\vec{b}) = (\nabla\times\vec{a})\cdot\vec{b} - \vec{a}\cdot(\nabla\times\vec{b})$
- $\nabla\times(\vec{a}\times\vec{b}) = (\nabla\cdot\vec{b})\vec{a} - (\nabla\cdot\vec{a})\vec{b} + [\vec{b}\cdot\nabla]\vec{a} - [\vec{a}\cdot\nabla]\vec{b}$
- $\nabla\times(\nabla\times\vec{a}) = \nabla(\nabla\cdot\vec{a}) - \nabla^2\vec{a}$ $\quad \cancel{\nabla^2\vec{a} = \frac{\partial^2 a_x}{\partial x} + +}\ \partial^2 a$

$$\nabla^2\vec{a} = \nabla^2 a_x\hat{i} + \nabla^2 a_y\hat{j} + \nabla^2 a_z\hat{k}$$

The operator $[\vec{a}\cdot\nabla]$

- this is a scalar operator with a scalar field gives a scalar field with a vector field gives a vector field

$$[\vec{a}\cdot\nabla] \equiv \left[a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}\right]$$

Maxwells Equations

$\text{div}\,\vec{D} = \rho$ $\quad \nabla\cdot\vec{D} = \rho$

$\text{div}\,\vec{B} = 0$ $\quad \nabla\cdot\vec{B} = 0$

$\text{curl}\,\vec{E} = -\frac{\partial}{\partial t}\vec{B}$ $\quad \nabla\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$

$\text{curl}\,\vec{H} = \vec{J} + \frac{\partial}{\partial t}\vec{D}$ $\quad \nabla\times\vec{H} = \vec{J} + \frac{\partial}{\partial t}\vec{D}$

$\left.\begin{array}{l}\vec{D} = \varepsilon\vec{E}\\ \vec{B} = \mu\vec{H}\end{array}\right\}$ similar to electric and magnetic fields but incorporate properties of the medium ε, μ

$\vec{E}$ - electric $\vec{B}$