$$\sinh x = \frac{e^{x}-e^{-x}}{2} \qquad \sinh^{-1}(x) = \ln\left(x \pm \sqrt{x^2+1}\right)$$

$$\cosh x = \frac{e^{x}+e^{-x}}{2} \qquad \cosh^{-1}(x) = \ln\left(x \pm \sqrt{x^2-1}\right)$$

$$\tanh x = \frac{1-e^{-2x}}{1+e^{-2x}} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \qquad \tanh^{-1}(x) = \ln\left(\sqrt{\frac{1+x}{1-x}}\right)$$

$$\cosh^2 x - \sinh^2 x \equiv 1 \qquad \cosh 2x \equiv \cosh^2 x + \sinh^2 x \qquad \sinh 2x \equiv 2\sinh x \cosh x$$

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \qquad \det M = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

minor: $\text{minor}(b_1) = \begin{pmatrix} a_1 & a_2 & a_3 \\ \cancel{b_1} & \cancel{b_2} & \cancel{b_3} \\ d_1 & c_2 & c_3 \end{pmatrix} = \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}$ signs: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

cofactor: $B_1 = -\text{minor}(b_1) = -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}$

adjoint: $\text{adj}(M) = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}^T = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$

$(MN)^{-1} = N^{-1}M^{-1}$

$AA^{-1} = A^{-1}A = I$

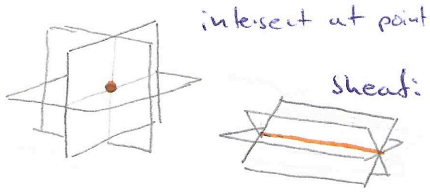
(if $A \neq$ singular)

$$M^{-1} = \frac{\text{adj}(M)}{\det(M)} = \frac{1}{\det(M)} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$$

given 3 equations in 3 unknowns: $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

intersect at point

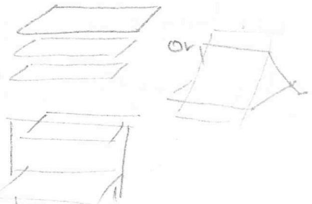
sheaf:



$\det M = 0?$ ⟶ M non-singular ⟶ unique solution

↘ M singular ⟶ equations consistant $\xrightarrow{\text{yes}}$ ∞ many solutions

$\searrow^{\text{no}}$ no solution



or

$M\vec{e} = \lambda\vec{e} \qquad (M - \lambda \hat{I})\vec{e} = 0$

finding eigenvectors:

1.) $\det(M - \lambda I) = 0$ 2.) find eigenvalues λ

3.) for each eigenvalue find eigenvector $\vec{e}$ by solving $(M - \lambda\hat{I})\vec{e} = 0$

- any position $\vec{p}$ can be expressed as $\alpha\vec{e}_1 + \beta\vec{e}_2$

then $M\vec{p} = M(\alpha\vec{e}_1 + \beta\vec{e}_2) = \alpha\lambda_1\vec{e}_1 + \beta\lambda_2\vec{e}_2$

- if M has eigenvalues + vectors $\vec{e}_1, \vec{e}_2$ and λ_1, λ_2

diagonalised to $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad Q = (\vec{e}_1 \ \vec{e}_2)$

then $M^n = QD^nQ^{-1} = Q\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}Q^{-1}$

- Cayley Hamilton theorem

$\det(M-\lambda I)=0 \Rightarrow a\lambda^2+b\lambda+c=0$ then $aM^2+bM+cI=0$

e.g. $M=\begin{pmatrix}4&2\\1&3\end{pmatrix}$ $\det(M-\lambda I)=0$
$\lambda^2-7\lambda+10=0$ $\quad M^2-7M+10I=0$ ← actually zero matrix $\vec{0}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$

can be used to create reccurence relationships
↳ every square matrix M satisfies its own characteristic equation

---

- for parametric equations $\frac{dx}{dt}=\dot{x}$ $\frac{dy}{dt}=\dot{y}$ $\quad \frac{d^2y}{dx^2}=\frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{(\dot{x})^3}$
- Maclaurian series: $f(x)=f(0)+f'(0)+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\dots$
↳ all usual series in data book also check valid domain of exponsion

---

- reduction formulae

$I_n=\int_0^1 x^n e^{2x}dx=\left[\frac{1}{2}x^n e^{2x}\right]_0^1-\int_0^1 nx^{n-1}\frac{1}{2}e^{2x}dx$ $\quad u=x^n$ $\quad \frac{du}{dx}=nx^{n-1}$ $\quad \frac{dv}{dx}=e^{2x}$ $\quad v=\frac{1}{2}e^{2x}$

$=\left(\frac{1}{2}e^2-0\right)-\frac{n}{2}\int_0^1 x^{n-1}e^{2x}dx$

$I_n=\frac{1}{2}e^2-\frac{n}{2}I_{n-1}$ (can be used to find I_4)

- rectangular approximations draw a sketch
- Arc length: - cartesian $\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ or $\frac{ds}{dy}=\sqrt{\left(\frac{dx}{dy}\right)^2+1}$

- polar $\frac{ds}{d\theta}=\sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}$

- parametric $\frac{ds}{dp}=\sqrt{\left(\frac{dx}{dp}\right)^2+\left(\frac{dy}{dp}\right)^2}$

- Volume, SA of revolution:
↳ about x axis $V=\int_a^b \pi y^2 dx$ $\quad SA=\int_a^b 2\pi y\, ds$
↳ about y axis $V=\int_a^b \pi x^2 dy$ $\quad SA=\int_a^b 2\pi x\, ds$

have to substitute ds

$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$ $\quad r = |z| = \sqrt{x^2 + y^2}$

$z^* = x - iy \quad zz^* = |z|$ $\quad \theta = \arg(x+iy) = \tan^{-1}\left(\frac{y}{a}\right)$

$-\pi < \theta \leq \pi$

$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$ $\rightarrow$ enlargement of z_1 by factor of $|z_2|$ and rotate through $\arg(z_2)$ (anti clockwise) or vice versa (commutative)

$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$

- De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \quad n \in \mathbb{Q}$

$\cos n\theta = \frac{z^n + z^{-n}}{2} \quad \sin n\theta = \frac{z^n - z^{-n}}{2i} \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

- sum of n^{th} roots of unity is 0 - root of unity denoted ω

$z = r(\cos\theta + i\sin\theta)$ $\quad n$th root of z $\sqrt[n]{z} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right)$

$\{k \in \mathbb{Z} \mid 0 \leq k < n\}$

$e^z = e^x(\cos y + i\sin y)$

$e^{z + 2\pi n i} = e^z$

$C = 1 + \binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + \binom{n}{3}\cos 3\theta \dots + \cos n\theta$

$S = \binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta + \binom{n}{3}\sin 2\theta \dots + \sin n\theta$

- identities:

do $C + iS$ and reduce as a sum

↳ 1) $\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$

then expand and equate Re or Im and use $\cos^2\theta + \sin^2\theta = 1$

↳ 2) $(2\cos\theta)^5 = (z + z^{-1})^5$ or $(2i\sin\theta)^5 = (z - z^{-1})^5$

then expand and reduce to $(z^n + z^{-n})$ to get $2\cos(n\theta)$ (or - for sin)

- 1^{st} order differential equations linear $\quad \frac{dy}{dx} + P(x)y = Q(x)$

integrating factor $R = e^{\int P(x)\,dx}$ ← +c can be omitted

then solve $\frac{d}{dx}(Ry) = RQ(x)$

- 2^{nd} order differential equation linear with constant coefficients

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ $\quad$ homogeneous if $f(x) = 0$

auxiliary equation: $a\lambda^2 + b\lambda + c = 0$

↳ 2 Re distinct roots λ_1, λ_2 $\quad y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$

↳ 2 repeated roots λ $\quad y(x) = (Ax + B)e^{\lambda x}$

↳ 2 complex roots $\lambda = \alpha \pm \beta i$ $\quad y(x) = e^{\alpha x}(A\sin(\beta x) + B\cos(\beta x))$

- SHM - described by DE $\frac{d^2x}{dt^2} + w^2 x = 0$

solution is $x = A\sin(wt) + B\cos(wt)$ period $= \frac{2\pi}{w}$

$= a\sin(wt + \varepsilon)$ amplitude $= \sqrt{A^2+B^2}$

$y(x) = A\sin(wx) + B\cos(wx)$

$= R\sin(wx + \varepsilon)$ $R = \sqrt{A^2+B^2}$ $\varepsilon = \tan^{-1}\left(\frac{B}{A}\right)$

- dampened Harmonic oscilator $\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + w^2x = 0$

$\alpha = 0$ - no damping $\lambda^2 + \alpha\lambda + w^2$

$b^2 - 4ac > 0$ over damped

$b^2 - 4w^2 = 0$ critical damping

$b^2 - 4ac < 0$ underdamped

- trial functions:

$y(x) = ax + b$ $y(x) = a\sin(px) + b\cos(px)$ $y(x) = ae^{px}$

- Transforming differential equations

↳ for $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ use $v = \frac{y}{x}$ so $y = vx$ $\frac{dy}{dx} = v + x\frac{dv}{dx}$

then solve using separation of variables

↳ for $ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = f(x)$ use $x = e^t$ $x\frac{dy}{dx} = \frac{dy}{dt}$

$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$

to transform into $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$

↳ for DE not containing y explicitly use:

$z = \frac{dy}{dx}$ $\frac{d^2y}{dx^2} = \frac{dz}{dx}$ transform into 1st order DE in z, x

↳ for DE not containing x explicitly use:

$z = \frac{dy}{dx}$ $\frac{d^2y}{dx^2} = z\frac{dz}{dy}$ transform into 1st order DE in z, y