Linear Algebra 2021

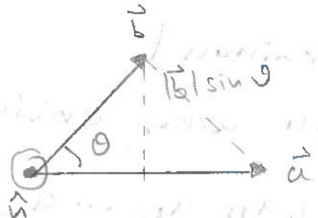
- dot product in $\mathbb{R}^n$ $\quad \vec a \cdot \vec b = \sum_{i=1}^{n} a_i b_i = |\vec a||\vec b| \cos\theta$
  - ↳ $\vec a$ along $\vec b$: $\vec a_{\parallel b} = (\vec a \cdot \hat b)\hat b$
  - ↳ $\vec a$ perpendicular to $\vec b$: $\vec a_{\perp b} = \vec a - (\vec a \cdot \hat b)\hat b$
  - ↳ commutative: $\vec a \cdot \vec b = \vec b \cdot \vec a$ $\quad$ distributive: $\vec a \cdot (\vec b + \vec c) = \vec a \cdot \vec b + \vec a \cdot \vec c$

- Flux
  - ↳ rate of flow of vector quantity through surface

area: $\vec A = A\,\hat a$ $\quad \hat a$ is normal to surface

$\phi = n \vec v \cdot \vec A$ $\quad n$ - number density $\quad \vec v$ - velocity

- cross product $\quad \vec a \times \vec b = |\vec a||\vec b| \sin\theta\, \hat n$



  - ↳ A of parallelogram is $|\vec a \times \vec b|$ so $\vec A = \vec a \times \vec b$

A triangle is $\frac{1}{2}|\vec a||\vec b|\sin\theta$

  - ↳ anti-commutative $\vec b \times \vec a = -\vec a \times \vec b$ since $\hat n$ reversed
  - distributive $\quad \vec a \times (\vec b + \vec c) = \vec a \times \vec b + \vec a \times \vec c$

- Planes
  - ↳ in $\mathbb{R}^3$ defined by: 3 points, 1 point + 2 vectors, normal vector + scalar

$\vec r = \vec r_0 + \lambda(\vec r_1 - \vec r_0) + \mu(\vec r_2 - \vec r_0)$ $\qquad \vec r \cdot \hat n = ax + by + cz = d$

$\oplus$ - into plane $\quad \odot$ - out of plane (imagine dart

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- Volume
  - ↳ vol of parallelepiped given by scalar triple product

$$V = |\vec c \cdot (\vec a \times \vec b)| = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

- vector triple product
  - ↳ decomposes $\vec a$ into components along $\vec b$ and $\vec c$

$$\vec a \times (\vec b \times \vec c) = \vec b(\vec a \cdot \vec c) - \vec c(\vec a \cdot \vec b)$$

- derivatives of vectors

in R^n $\frac{d\vec{a}}{ds} = \sum_{i=1}^{n} \frac{da_i}{ds}\hat{i}_i = \frac{da_1}{ds}\hat{\imath} + \frac{da_2}{ds}\hat{\jmath} + \frac{da_3}{ds}\hat{k} + \dots$

↳ $\frac{d\vec{a}}{dt} = \underbrace{\frac{da}{dt}\hat{a}}_{\text{along } \hat{a}} + \underbrace{a\vec{\omega}\times\hat{a}}_{\text{perpendicular } \hat{a}}$

↳ product rule: $\frac{d}{ds}\left(f(s)\,\vec{a}(s)\right) = f(s)\frac{d\vec{a}}{ds} + \frac{df}{ds}\vec{a}$

dot product rule: $\frac{d}{ds}\left(\vec{a}(s)\cdot\vec{b}(s)\right) = \vec{a}\cdot\frac{d\vec{b}}{ds} + \frac{d\vec{a}}{ds}\cdot\vec{b}$

cross product rule: $\frac{d}{ds}\left(\vec{a}(s)\times\vec{b}(s)\right) = \left(\frac{d\vec{a}}{ds}\times\vec{b}\right) + \left(\vec{a}\times\frac{d\vec{b}}{ds}\right)$

- determinant

↳ $n\times n$ array evaluated into a scalar

$$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_{ij} \quad \text{fixed } i$$
$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} A_{ij} \quad \text{fixed } j$$

A_{ij} is a minor $(-1)^{i+j}A_{ij}$ is a cofactor

↳ Laplace method $D_2 = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} a+j & b+k & c+l \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} j & k & l \\ d & e & f \\ g & h & i \end{vmatrix}$$

- properties of determinants

↳ swapping rows or columns changes sign

↳ cyclic permutation changes sign by $(-1)^{n-1}$

↳ transpose doesnt affect determinant

↳ any row or colum can be expanded or combined

↳ multiplying by scalar multiplies one row or column

↳ if any row or colum is all 0 det = 0
if any rows or columns are multiples then det = 0

↳ any multiple of row or column can be added to another without changing the determinant

- Triangular matrix
  ↳ matrix where terms above or below diagonal are all zeros
  ↳ determinant is the product of diagonal elements

- Cramer's rule

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad x = \frac{\Delta_1}{\Delta}$$

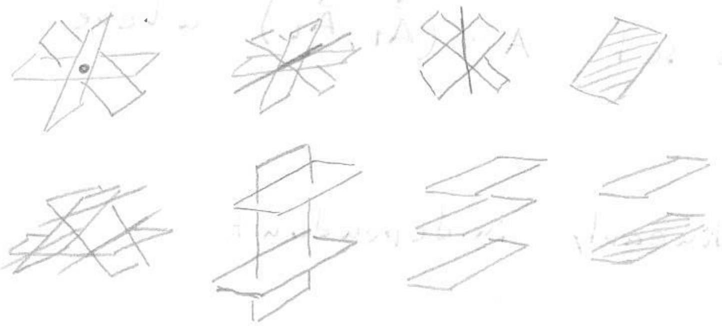
$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

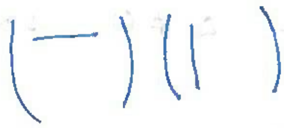
if $\Delta = 0 \quad \Delta_1 = \Delta_2 = \Delta_3 = 0$ ∞ solutions
$\Delta = 0 \quad \Delta_1 = \Delta_2 = \Delta_3 \neq 0$ no solutions

- Gaussian elimination
  ↳ augmented matrix

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right)$$

1.) interchange any rows
2.) multiply rows by constant
3.) add or subtract multiples of any rows





- matrix
  ↳ array of numbers $n \times m$
  ↳ addition: $A + B = a_{ij} + b_{ij}$ scalar multiplication: $\lambda A = \lambda a_{ij}$
  ↳ transpose: $A = a_{ij}$ $A^T = a_{ji}$ for non square are different shapes
  ↳ matrix multiplication: $A = a_{ij}$ $B = b_{ij}$ $AB = \sum_{j=1}^{j} a_{ij} b_{jk}$

non commutative: $AB \neq BA$ distributive: $(A+B)C = AC + BC$

associative: $A(BC) = (AB)C$ zero matrix: $0A = 0$

unit matrix: $A \mathbb{I}_j = A$ $\mathbb{I}_i A = A$

- inverse of 2x2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- adjoint of a matrix

$$\operatorname{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \operatorname{adj}(A) A = \det A \, \mathbb{I}$$

- inverse by gaussian elimination

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{form} \quad \left(\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right)$$

- dot product $\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$

- Basis set
  - ↳ set B of vectors in vector space V is a basis set if every element of V can be written as a unique finite linear combination of elements of B
  - ↳ the coefficients are the coordinates

$$A\vec{x} = \vec{b} \Rightarrow \vec{A}_1 x + \vec{A}_2 y = \vec{b} \Rightarrow \begin{matrix} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{21}y = b_2 \end{matrix}$$

so $\vec{b}$ is decomposed into basis set $A = (\vec{A}_1, \vec{A}_2)$ where $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ are the coordinates

↳ can be done if $\Delta \neq 0$ so linearly independent

- Gram - Schmidt process

1.) start with basis set check linear dependence

2.) make normal vector $\hat{e}_1 = \frac{\vec{V}_1}{|\vec{V}_1|}$

3.) find normal component of $\vec{V}_2$ $\quad \vec{V}_2' = \vec{V}_2 - (\vec{V}_2 \cdot \hat{e}_1)\hat{e}_1$

4.) normalise $\vec{V}_2'$ $\quad \hat{e}_2 = \frac{\vec{V}_2'}{|\vec{V}_2'|} = \frac{\vec{V}_2 - (\vec{V}_2 \cdot \hat{e}_1)\hat{e}_1}{|\vec{V}_2'|}$

- Decompose vector into orthonormal basis

$x_i = \hat{e}_i \cdot \vec{V}$ $\quad$ $\vec{V}$ - vector being decomposed

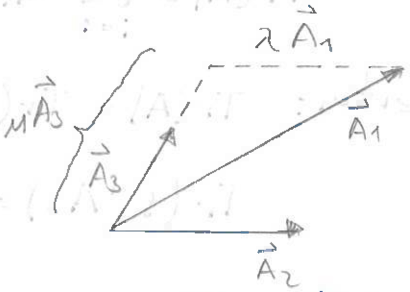
$\hat{e}_i$ - orthonormal basis vector

x_i - coefficient corresponding to $\hat{e}_i$

- homogeneous equation

$A\vec{x} = \vec{b}$ if $\Delta \neq 0$ trivial solution

$A\vec{x} = \vec{0}$ if $\Delta = 0$ non-trivial solutions



in $\mathbb{R}^3$ $\quad 0 = x_1\vec{A}_1 + x_2\vec{A}_2 + x_3\vec{A}_3$

$$\vec{A}_1 = -\frac{x_2}{x_1}\vec{A}_2 + -\frac{x_3}{x_1}\vec{A}_3$$

$\vec{A}_1 = \lambda\vec{A}_2 + \mu\vec{A}_3$ so if homogeneous then the vectors are linearly dependent

↳ in $\mathbb{R}^3$ linearly dependent vectors are called coplanar

↳ n linearly dependent vectors span space $\mathbb{R}^{n-n}$ in $\mathbb{R}^n$

- eigenvalues

special case of $A\vec{x} = \vec{b}$ where $A\vec{x} = \lambda\vec{x}$

↳ characteristic eqn: $\det(A - \lambda\mathbb{I}) = 0$

↳ homogeneous eqn: $(A - \lambda\mathbb{I})\vec{e} = \vec{0}$

↳ eigen vectors invariant under A · scaled by λ

- singular matrix reduces dimension $\Rightarrow$ loss of information

- Similarity transformation

↳ Λ diagonal matrix with eigenvalues

$$S^{-1}AS = \Lambda \quad \Rightarrow \quad A = S\Lambda S^{-1}$$

↳ $S^{-1}AS$ decomposes A into basis corresponding to eigenvectors of

$\det(\Lambda) = \det(A)$ so $\det(A) = \prod_{i=1}^{n} \lambda_i$

$$A^n = S\Lambda^n S^{-1}$$

- Trace $Tr(A) = \sum_{i=1}^{n} a_{ii}$ sum of diagonal elements

↳ properties: $Tr(A) = Tr(A^{-1})$ $Tr(AB) = Tr(BA)$

$Tr(S^{-1}AS) = Tr(\Lambda) = Tr(A)$

- symmetric matrix: $A = A^T$

↳ products of symmetric matrices commute $AB = BA$

- Hermitian matrix: $A = A^+ = A^{T*}$

- Skew symmetric matrix: $A = -A^T$

↳ has to have all 0 on the diagonal since $a_{ij} = -a_{ji} = 0$

- normal matrix: $AA^T = AA^{T*}$

↳ happens only if eigenvectors of A form normal basis set

↳ all diagonal, symmetric, skew-symmetric metrices are normal

- linear transformation: $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$

- affine transformation: linear transformation + translation

- matrix transformation

↳ transform from $\mathbb{R}^n \rightarrow \mathbb{R}^n$

↳ defined by its effect on basis set

↳ determinant is ratio of areas/volumes $\det(A) = \frac{SA_I}{SA_0}$

↳ combinations $A_2 A_1 X = A_{21} X$ $A_{21} = A_2 A_1$

- transformations in 3D

↳ if z component unchanged (in x-y plane) just expand the 2x2 matrix

- 2D rotation $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

- 3D rotation

$$A_{xy} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A_{yz} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad A_{xz} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

↳ in 3D succesive rotations aren't comutative