- Hilberts space - abstract vector space with finite or infinite dimensionality

- Bra - Ket vectors: unit 4 vector usually

$|A\rangle + |B\rangle = |C\rangle$ $|A\rangle + |B\rangle = |B\rangle + |A\rangle$ $(|A\rangle + |B\rangle) + |C\rangle = |A\rangle + (|B\rangle + |C\rangle)$

$|A\rangle + (-|A\rangle) = 0$ $|zA\rangle = z|A\rangle$ $z(|A\rangle + |B\rangle) = z|A\rangle + z|B\rangle$

$(z+w)|A\rangle = z|A\rangle + w|A\rangle$

$z|A\rangle \Rightarrow \langle A|z^*$ $|A\rangle = \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}$ $\langle A| = (\alpha_1^* \ \alpha_2^* \ \alpha_3^* \ \alpha_4^*)$

- inner product = $\langle B|A\rangle = (\beta_1^* \ \beta_2^* \ \beta_3^* \ \beta_4^*)\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix} = \alpha_1\beta_1^* + \alpha_2\beta_2^* + \alpha_3\beta_3^* + \alpha_4\beta_4^*$

$\langle B|A\rangle = \langle A|B\rangle^*$

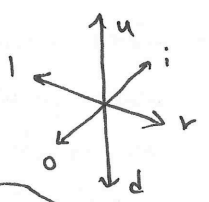
$\langle C|(|A\rangle + |B\rangle) = \langle C|A\rangle + \langle C|B\rangle$

- normalized vector - of length 1 so gives inner product of 1
  - $\langle A|A\rangle = 1$ if A normalized

- orthogonal vectors - at right angles (perpendicular) to eachother
  - $\langle B|A\rangle = 0$ if A, B orthogonal

- orthogonal normalized bases - orthonormal bases
  - orthogonal + normalized used to orient in hilbert space

eg $\langle u|d\rangle = 0$
$\langle l|r\rangle = 0$
$\langle i|o\rangle = 0$



- linear operators - expressed by matrixes act on vectors

$M|A\rangle = |B\rangle$ $Mz|A\rangle = z|B\rangle$ $M(|A\rangle + |B\rangle) = M|A\rangle + M|B\rangle$

- eigenvalues and eigenvectors $|\lambda\rangle$ and λ $M|\lambda\rangle = \lambda|\lambda\rangle$

- Hermetian conjugation

$M|A\rangle = |B\rangle$ $\langle A|M^\dagger = \langle B|$ and $Mz|A\rangle = z|B\rangle$ $\langle A|z^*M^\dagger = \langle B|z^*$

$M^\dagger$ - hermitian conjugate

hermitian operator if $M = M^\dagger$

$M^\dagger = (M^T)^*$ $M = \begin{pmatrix}m_{11} & m_{12} & m_{13}\\ m_{21} & m_{22} & m_{23}\\ m_{31} & m_{32} & m_{33}\end{pmatrix}$ → called transposing - reflect. over diagonal so $m_{12} \Rightarrow m_{21}$

$M^T = \begin{pmatrix}m_{11} & m_{21} & m_{31}\\ m_{12} & m_{22} & m_{32}\\ m_{13} & m_{23} & m_{33}\end{pmatrix}$ → find complex conjugate of each component so $m_{12} \Rightarrow m_{12}^*$

$M^\dagger = \begin{pmatrix}m_{11}^* & m_{21}^* & m_{31}^*\\ m_{12}^* & m_{22}^* & m_{32}^*\\ m_{13}^* & m_{23}^* & m_{33}^*\end{pmatrix}$

- probability of a state

given state vector $|A\rangle$

$|A\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle$

$= \alpha_u \begin{pmatrix}1\\0\end{pmatrix} + \alpha_d \begin{pmatrix}0\\1\end{pmatrix}$

$= \begin{pmatrix}\alpha_u\\ \alpha_d\end{pmatrix}$ in terms of orthonormal vectors

$\alpha_u = \langle u|A\rangle$

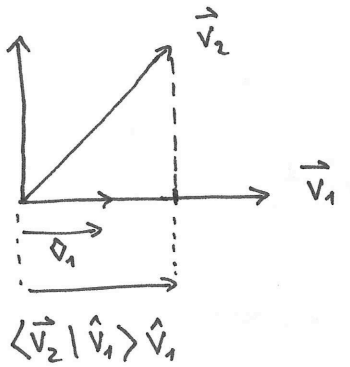
$\alpha_d = \langle d|A\rangle$

$P_{(\text{spin up})} = P_u = \alpha_u^2 = \langle A|u\rangle\langle u|A\rangle$

$P_{(\text{spin down})} = P_d = \alpha_d^2 = \langle A|d\rangle\langle d|A\rangle$

- Gram - Schmidt procedure

given $\vec{V}_1$ and $\vec{V}_2$ form orthonormal bases



$\hat{V}_1 = \frac{\vec{V}_1}{|\hat{V}_1|}$

$\langle\vec{V}_2|\hat{V}_1\rangle\hat{V}_1$ - component of $\vec{V}_2$ along $\vec{V}_1$ so "projected on $\vec{V}_1$"

$\vec{V}_{2\perp}$ - component to $\vec{V}_2$ orthogonal to $\vec{V}_1$

$\vec{V}_{2\perp} = \vec{V}_2 - \langle\vec{V}_2|\hat{V}_1\rangle\hat{V}_1$ since $\vec{V}_2 = \vec{V}_{2\parallel} + \vec{V}_{2\perp}$ parallel + normal

$\hat{V}_2 = \frac{\vec{V}_{2\perp}}{|\vec{V}_{2\perp}|}$

- observables are represented by Hermitian linear operators

the corresponding measured quantity is λ_i (eigenvalue)

and ~~the state is $|\lambda_i\rangle$ so the eigenvector~~

$|\lambda_i\rangle$ is the determined state by experiment

- state prepared in state $|A\rangle$ observing observable M

$P(\lambda_i) = |\langle A|\lambda_i\rangle|^2$ where $M|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$

- spin represented by operator $\hat{\sigma} = \begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{pmatrix} = \begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}$ in spherical coordinates

eg $\sigma_z|u\rangle$ and $|d\rangle$

$\sigma_z = +1$ $\sigma_z = -1$

use to find θ and ϕ

then you can find other components such as σ_x, σ_y

$|u\rangle$ and $|d\rangle$ are eigenvectors of the spin operator

$\sigma_z|u\rangle = (+1)|u\rangle$ $\sigma_z|d\rangle = (-1)|d\rangle$

both are orthogonal $\langle u|d\rangle = 0$

- pauli matrices for spin

$\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$ $\sigma_z \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

eigenvectors:

$|\lambda_1\rangle = \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}$ $|\lambda_2\rangle = \begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}$

- Spin polarization principle (single spin)
  - any state of a spin is an eigenvector of some component of the spin
  - ∴ given state $|A\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle$
  there exists a ~~space~~ direction $\hat{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$ for which $\vec{\sigma}\cdot\vec{n}|A\rangle = |A\rangle$
  so $\vec{\sigma}\cdot\vec{n} = \lambda$ of $|A\rangle$ and $= +1$
  as a result $P_n = \langle\vec{\sigma}\cdot\vec{n}\rangle^2 = +1$
  - $\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 = 1$
  $\vec{\sigma}\cdot\vec{n} = \vec{\sigma}_n$ – component of spin $\vec{\sigma}$ along direction of $\hat{n}$

- Unitarity – condition that time evolution of quantum state according to Schrödingers equation is represented by unitary operator (time developement operator)

$|\psi(t)\rangle = U(t)|\psi(0)\rangle$ $U(t)$ – time developement operator

- unitary operator $U^\dagger(t)\,U(t) = I$ I – unit operator

- states are distinguishable if they are orthogonal (there is an experiment to precisely tell them apart)
  if $|\psi(0)\rangle, |\phi(0)\rangle$ are two orthogonal states
  therefore $\langle\psi(0)|\phi(0)\rangle = 0$
  they will continue to be orthogonal for all t $\langle\psi(t)|\phi(t)\rangle = 0$

- characteristic equation for finding λ and $|\lambda\rangle$

$\det(M - \lambda I) = 0$ or $(M-\lambda I)|\lambda\rangle = 0$

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

- time developement also continuous so changes smoothly

Hamiltonian – linear operator H

time independant schrödinger equation: $\hbar\frac{\partial|\psi\rangle}{\partial t} = -iH|\psi\rangle$

$\hbar \approx 1.05\times10^{-34}$ kg m² s⁻¹

- average or expected value $\langle L\rangle = \langle A|L|A\rangle$ where L – linear operator, $|A\rangle$ – state vector
$= \sum_i (\alpha_i^*\alpha_i)\lambda_i$

these factors can be ignored as they cancel out

$\langle L\rangle = \langle A|e^{-i\theta}L\,e^{i\theta}|A\rangle$
$= \langle A|L\,e^{0}|A\rangle$
$= \langle A|L|A\rangle$

expectation value at time t
$= \langle\psi(t)|L|\psi(t)\rangle$

- commutation: - commutator of L with M:

$$[L,M] = LM - ML \qquad [L,M] = -[M,L]$$

$$\frac{d\langle E\rangle}{dt} = -\frac{i}{\hbar}\langle [L,H]\rangle \qquad \frac{dL}{dt} = -\frac{i}{\hbar}[L,H]$$

quitting the angle brackets $\langle\rangle$

syetrical to classical mechanics where $\frac{dF}{dt} = \{F,H\}$

- converting quantum commutators to poisson brackets: $[F,G] = i\hbar\{F,G\}$

how we can do in this case $[L,H] = i\hbar\{L,H\}$ where L is an observable

$$\frac{dL}{dt} = -\frac{i}{\hbar}(i\hbar\{L,H\})$$

$$= \{L,H\}$$

- conservation

if quantity or operator commutes with hamiltonian H it is conserved $[Q,H]=0$ this works for all powers of Q so Q^n therefore ~~if quantity~~ all functions of Q^n conserved

↳ if $[Q^n,H]=0$ then quantity tied to observable Q then it is conserved

eg H - represents energy $[H,H]=0$ ∴ Energy conserved

- spin in the magnetic field

energy proportional dot product of spin and magnetic field

$$H \propto \vec{\sigma}\cdot\vec{B} = \sigma_x B_x + \sigma_y B_y + \sigma_z B_z$$

field lies along z axis so $B_x=0$ $B_y=0$ ∴ $H \propto \sigma_z B_z$

$$H = \frac{\hbar w}{2}\sigma_z$$ where w contains all constants for simplicity

$$\langle\dot{\sigma}_x\rangle = -\frac{i}{\hbar}\langle[\sigma_x,H]\rangle = -\frac{iw}{2}\langle[\sigma_x,\sigma_z]\rangle = -w\langle\sigma_y\rangle$$

$$\langle\dot{\sigma}_y\rangle = -\frac{i}{\hbar}\langle[\sigma_y,H]\rangle = -\frac{iw}{2}\langle[\sigma_y,\sigma_z]\rangle = w\langle\sigma_x\rangle$$

$$\langle\dot{\sigma}_z\rangle = -\frac{i}{\hbar}\langle[\sigma_z,H]\rangle = -\frac{iw}{2}\langle[\sigma_z,\sigma_z]\rangle = 0$$ since they commute

use pauli matrices for spin to compute commutators

so $[\sigma_x,\sigma_z] = 2i\sigma_z$ $[\sigma_y,\sigma_z] = 2i\sigma_x$ $[\sigma_z,\sigma_z] = 2i\sigma_y$

again $x \to y \to z$

- solving schrödinger equation

$$i\hbar \frac{\partial \Psi(x)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x)}{\partial x^2} + U(x)\,\Psi(x)$$

↳ procedure :

1.) find and express hamiltonian $H =$

2.) prepare an initial state $|\Psi(0)\rangle$

3.) solve time independant equation to get λ_i and $|\lambda_i\rangle$

$$H|E_i\rangle = E_i|E_i\rangle \qquad \text{or } E_i \text{ and } |E_i\rangle$$

4.) use $|\Psi(0)\rangle$ and $E_i, |E_i\rangle$ to calculate initial coefficients $\alpha_i(0)$

$$\alpha_i(0) = \langle E_i|\Psi(0)\rangle$$

5.) rewrite $|\Psi(0)\rangle$ in terms of eigenvectors $|E_i\rangle$ and initial coefficients $\alpha_i(0)$

$$|\Psi(0)\rangle = \sum_i \alpha_i(0)|E_i\rangle \qquad \text{check if still makes sense}$$

6.) replace initial with time dependant $0 \Rightarrow t$

$$|\Psi(t)\rangle = \sum_i \alpha_i(t)|E_i\rangle$$

(eigenvalue of $|E_i\rangle$)

7.) using equation below express $\alpha_i(t)$ in terms of α_i, E_i and, t

~~$\alpha_i(0) = \langle E_i|\Psi(0)\rangle$~~ so $|\Psi(t)\rangle = \sum_i \alpha_i(0)\, e^{-\frac{i}{\hbar} E_i t}$

$$\alpha_i(t) = \alpha_i(0)\, e^{-\frac{i}{\hbar} E_i t}$$

8.) values can be predicted using eigenvalues and eigenvectors

suppose observing observable L with eigenvalue λ and vector $|\lambda\rangle$

$$P_\lambda(t) = |\langle\lambda|\Psi(t)\rangle|^2 = \langle\lambda|\Psi(t)\rangle\langle\Psi(t)|\lambda\rangle$$

- solving tive dependant Schrödinger equation short:

1.) find or get Hamiltonian

2.) prepare initial state $|\Psi(0)\rangle$

3.) solve $H|E_i\rangle = E_i|E_i\rangle$ so find eigenvalue of state vectors

4.) find initial coefficients $\alpha_i(0) = \langle E_i|\Psi(0)\rangle$

5.) substitute into $|\Psi(0)\rangle = \sum_j \alpha_j(0)|E_j\rangle$ α_i is coefficient $|E_i\rangle$ is state vector

6.) solve $\alpha_i(t) = \alpha_i(0) e^{-\frac{i}{\hbar} E_i t}$

7.) substitute into $|\Psi(t)\rangle = \sum_i \alpha_i(t)|E_i\rangle$

8.) simplify if necessary. check using $t = 0$ if $|\Psi(0)\rangle$ is obtained.

- note - a measurement leaves the system in an eigenstate (state of a single eigenvector) corresponding to the last value (eigenvalue) measured
  - eg if λ_i measured state is now $|\lambda_i\rangle$

- state of multiple measurables
  - consider 2 independent spins. each has an observable, eigenvalue, eigenvector corresponding to hermitian operator L and M
  - after a measurement system is in a state which is both an eigenvector of L and M called simultaneous eigenvector of operators

state vector of each spin: $L|\lambda\rangle = \lambda|\lambda\rangle$ ~~$M|\lambda\rangle$~~ $= M|\mu\rangle = \mu|\mu\rangle$

bases of state vectors which are simultaneous eigenvectors of both ~~M~~ M, L

$LM|\lambda,\mu\rangle = \lambda\mu|\lambda,\mu\rangle$ in order for this to be possible L, M must commute

since order of multiplication of eigenvalues doesnt matter

$\therefore LM|\lambda,\mu\rangle = ML|\lambda,\mu\rangle$ so ~~$[L,M]|\lambda,\mu\rangle$~~ $[L,M]|\lambda,\mu\rangle = 0$ since $[L,M] = 0$

- the condition for 2 observables to be simultaneously measured is for their hermitian operators to commute
- if 2 hermitian operators commute there exists a complete basis of simultaneous eigenvectors (and eigenvalues) of the observables
- for higher n of observables they must all commute with eachother they are called complete set of commuting observables

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- wave function

suppose bases for some quantum system we have:
- orthonormal basis vectors $|a,b,c\ldots\rangle$
- eigenvalues $a, b, c\ldots$
- complete set of counting observables $A, B, C\ldots$
- arbitrary state vector $|\Psi\rangle$

since $|a,b,c\ldots\rangle$ orthonormal

$|\Psi\rangle$ can be expressed in term of $|a,b,c\ldots\rangle$

$$|\Psi\rangle = \sum_{a,b,c\ldots} \psi(a,b,c\ldots)|a,b,c\ldots\rangle$$

$\psi(a,b,c\ldots)$ are coefficients for expansion

also each coefficient = inner product of $|\Psi\rangle$ with basis vector

so $\psi(a) = \langle a|\Psi\rangle$ for all: $\psi(a,b,c\ldots) = \langle a,b,c\ldots|\Psi\rangle$

$\psi(a,b,c\ldots)$ - is the wave function

$P(a) = \psi^*(a)\,\psi(a)$ for all $P(a,b,c\ldots) = \psi^*(a,b,c\ldots)\psi(a,b,c\ldots)$

wave function in b_z $\psi(u) = \langle u|\Psi\rangle$ $\psi(d) = \langle d|\Psi\rangle$

P must allways sum to 1 $\sum_{a,b,c\ldots} \psi^*(a,b,c\ldots)\,\psi(a,b,c\ldots) = 1$

$$\sum_i \psi^*(i)\,\psi(i) = 1$$

- Any 2×2 Hermitian matrix representing operator can be written as sum of 4 terms with real coefficients

$$L = a\sigma_x + b\sigma_y + c\sigma_z + d\,\mathbb{I} \qquad a,b,c,d \in \mathbb{R}$$

$\sigma_x, \sigma_y, \sigma_z$ - pauli matrices

$\mathbb{I}$ - unity operator $= \begin{pmatrix}1&0\\0&1\end{pmatrix}$

- Spin components can not be measured simultaneously since don't commute

$$[\sigma_x,\sigma_y] = 2i\sigma_z \quad [\sigma_y,\sigma_z] = 2i\sigma_x \quad [\sigma_z,\sigma_x] = 2i\sigma_y$$

$\hat{z}\leftarrow \begin{matrix}x\\y\end{matrix}$

$\therefore [\sigma_x,\sigma_y] \neq 0$

- Uncertainty principle

↳ if hermitian operators of 2 observables do not commute there must be uncertainty in measurement of one/other/both

given state $|\psi\rangle$ and observable A with eigenvalue a

$P(a)$ is probability of observing a $\quad \langle\psi|A|\psi\rangle = \sum_a a P(a)$

since expected val = Σval x probability of value

- uncertainty in A is its standard deviation

begin by defining $\bar{A}$ $\quad \bar{A} = A - \langle\psi|A|\psi\rangle\mathbb{I}$ so $\bar{A}$ = A - expectation value x $\mathbb{I}$

$= A - \langle A\rangle\mathbb{I}$

we use $\mathbb{I}$ as unit operator to subtract $\langle A\rangle$ which is a single $\mathbb{R}$ value

this also shifts eigenvalues of A $\quad \bar{a} = a - \langle\psi|A|\psi\rangle\mathbb{I}$

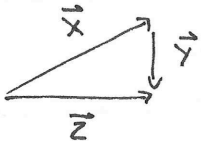
$= a - \langle A\rangle\mathbb{I}$

if uncertainty = ~~variatio~~ standard deviation = ΔA $\quad$ variance = $(\Delta A)^2$

$$(\Delta A)^2 = \sum_a \bar{a}^2 P(a) = \sum_a (a - \langle A\rangle\mathbb{I})^2 P(a)$$

which can be written as $(\Delta A)^2 = \langle\psi|\bar{A}^2|\psi\rangle$ $\quad$ if $\langle A\rangle = 0$ $(\Delta A)^2 = \langle\psi|A^2|\psi\rangle$

- Cauchy - Schwarz Inequality



$|x+y|$ is shortest $|z|$ can be

in real vector space $|x| + |y| \geq |x+y|$

which gives $|x||y| \geq |x\cdot y|$

↳ length of vector = ~~its~~ square root of its inner product with itself

$$|\vec{M}| = \sqrt{\langle M|M\rangle}$$

Cauchy - Schwarz inequality: $|\vec{x}|^2|\vec{y}|^2 \geq |\vec{x}\cdot\vec{y}|^2$

↳ the form used in uncertainty principle:

$$2|\vec{x}||\vec{y}| \geq |\langle x|y\rangle + \langle y|x\rangle|$$

- General Uncertainty principle

let $|\psi\rangle$ be any state A, B are any 2 observables

define $|x\rangle = A|\psi\rangle$ $|y\rangle = iB|\psi\rangle$ (dont fuget i !)

substitute $|x\rangle$ and $|y\rangle$ into cauchy schwarz inequality

$$2|x||y| \geq |\langle x|y\rangle + \langle y|x\rangle|$$

$$2\sqrt{\langle A^2\rangle}\sqrt{\langle B^2\rangle} \geq |\langle\psi|AB|\psi\rangle - \langle\psi|BA|\psi\rangle|$$

$$2\sqrt{\langle A^2\rangle\langle B^2\rangle} \geq |\langle\psi|[A,B]|\psi\rangle| \quad \text{using notation of comutators}$$

$$\Delta\bar{A}\,\Delta\bar{B} \geq \frac{1}{2}|\langle\psi|[\bar{A},\bar{B}]|\psi\rangle|$$

$$\Delta A\,\Delta B \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle|$$

product of uncertanties can not be smaller than $\frac{1}{2}$ magnitude of expectation value of comutator of observables

Heisenbergs uncertainty principle is product of uncertainty of position (q) and momentum (p) is greater or equal to $\frac{1}{2}$ plancks constat ($\frac{1}{2}\hbar$)

- states of a combined system

2 independant spins alice and bob can have states $|u\rangle$ and $|d\rangle$ each

they exist in state space S_A and S_B

we define the state space of both alice and bob S_{AB}

$S_{AB} = S_A \otimes S_B$ where RH is the tensor product

the orthonormal basis of S_{AB} are $|uu\rangle, |ud\rangle, |du\rangle, |dd\rangle$

- statistical correlation - shows how correlated observations of alice and bob are

if alice and bob toss a coin H = +1 T = -1 averges $\langle\delta_A\rangle = 0$ $\langle\delta_B\rangle = 0$

if system is correlated eg for every H alice gets bob gets a T

the product $\delta_A\delta_B = -1$ all ways so $\langle\delta_A\delta_B\rangle = -1$

this would be perfect (100%) correlation

statistical correlation $= \langle\delta_A\delta_B\rangle - \langle\delta_A\rangle\langle\delta_B\rangle$

if 2 variables compeletly uncorrelated $P(a,b) = P_A(a)\,P_B(b)$ so factorise

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- Product states

$$|\text{product state}\rangle = \left(\alpha_u|u\rangle + \alpha_d|d\rangle\right) \otimes \left(\beta_u|u\rangle + \beta_d|d\rangle\right)$$

$$= \alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle$$

if state vectors for alice ($|u\rangle, |d\rangle$) and bob ($|u\rangle, |d\rangle$)

are normalized the product state is automatically normalized aswell

it took 2 $\mathbb{R}$ parameters for a single ~~state~~ spin system now it is 4 $\mathbb{R}$ parameters

$(\alpha_u, \alpha_d, \beta_u, \beta_d)$

- entanglement is not "all or nothing" proposition there are levels of entanglement
- singlet and triplet states are most entangled states possible and can not be written in product ~~form~~ state

$$|sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle) \qquad |T_1\rangle = \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)$$

$$|T_2\rangle = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) \qquad |T_3\rangle = \frac{1}{\sqrt{2}}(|du u\rangle - |dd\rangle)$$

- an entangled state gives complete description of combined system no more can be known
- in maximally entangled state nothing is known about the individual systems even if you know entangled state

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- Tensor product operators

$$\sigma_z |du\rangle = -|du\rangle$$

$$(\sigma_z \otimes I)(|d\rangle \otimes |u\rangle) = (\sigma_z|d\rangle \otimes I|u\rangle) = (-|d\rangle \otimes |u\rangle)$$

I is unit operator so σ_z only affects ~~bob's~~ alices (first) half likewise T_z would affect bobs halfe only

- spin-polarization principle – not all expectation values can be 0

$$-\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 = 1$$

applies to all entangled states except singlet state $|sing\rangle$

sing goes to 0 for $|sing\rangle$ ~~$\langle\sigma_z\rangle$~~ $\langle\sigma_z\rangle = \langle\sigma_y\rangle = \langle\sigma_x\rangle = 0$

- $\vec{\sigma} \cdot \vec{T} = \sigma_x T_x + \sigma_y T_y + \sigma_z T_z$

$$= (\sigma_x \otimes I)(I \otimes T_x) + (\sigma_y \otimes I)(I \otimes T_y) + (\sigma_z \otimes I)(I \otimes T_z)$$

- for tensor products

$$\alpha = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad \beta = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \qquad \alpha \otimes \beta = \begin{pmatrix} a_1 \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} & a_2 \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \\ a_3 \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} & a_4 \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \end{pmatrix}$$

if α has N_α dimensions and β has N_β dimensions — $\alpha \otimes \beta$ has $N_\alpha N_\beta$

eg α: 3×4 β: 2×5 — $\alpha \otimes \beta$: $12 \times 10 = 120$

- can be applied for all eve solv schrödinger tise deedt

- given matrix M and basis vectors j,k we can calculate the matrix entries of M with respect to the basis

$$M_{jk} = \langle j|M|k\rangle \qquad \text{eg.) } M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad M_{jk} = \langle i|M|k\rangle = (1\ 0)\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|j\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |k\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad = (1\ 2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

- Tensor product matrices:

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}$$

eg.) $|uu\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

- outer product gives linear operator

outer product: $|\psi\rangle\langle\phi|$ if it acts on $|A\rangle$: $|\psi\rangle\langle\phi|\,|A\rangle \equiv |\psi\rangle\langle\phi|A\rangle$

if it acts on $\langle B|$: $\langle B|\,|\psi\rangle\langle\phi| \equiv \langle B|\psi\rangle\langle\phi|$

- outer product of same bra and ket = projection operator

$|\psi\rangle\langle\psi|$ projects vector onto direction of $|\psi\rangle$

- projection operators: - are Hermitian $(|\psi\rangle\langle\psi|)^\dagger = |\psi\rangle\langle\psi|$
  - any vector orthogonal to $|\psi\rangle$ is an eigenvector with eigenvalue of 0
  - vector $|\psi\rangle$ is an eigenvector with eigenvalue of 1
  - square of projection operator is itself $(|\psi\rangle\langle\psi|)^2 = |\psi\rangle\langle\psi|$
  - ~~trace of an operator is 1~~
  - Trace of the projection operator is 1 $\mathrm{Tr}\,|\psi\rangle\langle\psi| = 1$

$\sum_i |i\rangle\langle i| = I$ ⟶ - sum of projection operators of basis system gives Identity operator I

- Trace is defined as the sum of matrixes diagonal entries

(\) eg.) $\mathrm{Tr}\begin{pmatrix} 6 & 5 & 3 \\ 4 & 2 & 0 \\ 1 & 0 & 5 \end{pmatrix} = 6+2+5 = 13$

- Trace of a hermitian operator is ~~it's~~ the sum of its eigenvalues

~~Tr(M)~~ $\mathrm{Tr}M = \sum_i \lambda_i$

- expectation value of any observable L in state $|\psi\rangle$ can be calculated as:

$$\langle\psi|L|\psi\rangle = \mathrm{Tr}|\psi\rangle\langle\psi|L$$

- ~~Pro:~~ Density matrices

if alice prepares state $|\psi\rangle$ expectation value = $\mathrm{Tr}|\psi\rangle\langle\psi|L$

if alice prepares state $|\phi\rangle$ expectation value = $\mathrm{Tr}|\phi\rangle\langle\phi|L$

if 50% chce $|\psi\rangle$ and 50% $|\phi\rangle$ expectation value = $\frac{1}{2}\mathrm{Tr}|\psi\rangle\langle\psi|L + \frac{1}{2}\mathrm{Tr}|\phi\rangle\langle\phi|L$

and density matrix would be = $\rho = \frac{1}{2}|\psi\rangle\langle\psi| + \frac{1}{2}|\phi\rangle\langle\phi|$

So expectation value $\langle L\rangle = \mathrm{Tr}\,\rho L$

- density matrix $\rho = P_1|\phi_1\rangle\langle\phi_1| + P_2|\phi_2\rangle\langle\phi_2| + P_3|\phi_3\rangle\langle\phi_3| + \ldots$

$$\rho = \sum_i \left(P_i|\phi_i\rangle\langle\phi_i|\right)$$

density matrix is an operator until basis are chosen

eg $\rho_{aa'} = \langle a|\rho|a'\rangle \qquad \langle L\rangle = \sum_{aa'} L_{a',a}\rho_{a,a'}$

- density matrices and entanglement

to calculate Alices density matrix ρ we might need the full wave function $\psi(a,b)$ including bobs halve. once we know ρ we can forget where we obtained it from and use it to calculate anything about Alices observations

eg to calculate $P(a)$ so alices state to be a

start $P(a,b) = \psi^*(a,b)\psi(a,b)$ by rules of probability summing over b gives $P(a)$

$$P(a) = \sum_b P(a,b) = \sum_b \psi^*(a,b)\psi(a,b) \rightarrow \text{this is the diagonal entry in the density matrix}$$

$$= \rho_{aa}$$

- properties of density matrices

↳ Hermitian $\rho_{aa'} = \rho^*_{a'a}$ $\left(\rho^\dagger_{aa'} = \rho_{a'a}\right)$ transpose just exchge entries in subscript

↳ Trace of density matrix is 1 $Tr(\rho) = 1$ (sum of probabilities must = 1)

↳ eigenvalues of ρ are all +ve and lie between 0 and 1

it follows if one $\lambda = 1$ others must be 0

(since sum of probability = 1 and one probability is 1 others = 0)

↳ for a pure state $\rho^2 = \rho$ $Tr(\rho^2) = 1$ (pure - unentangled)

↳ for a mixed state $\rho^2 \neq \rho$ $Tr(\rho^2) < 1$ (mixed - entangled)

- Trace of product of 2 matrices is commutative

$$Tr(AB) = Tr(BA) \qquad AB \neq BA$$

- Calculating alices ρ (a and a' are 2 stks fr alice)

$\rho_{a'a} = \sum_b \psi^*(a,b)\psi(a',b)$ consider state vector $|\psi\rangle = \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)$

recall basis vector coefficients $\psi(u,u) = 0 \quad \psi(u,d) = \frac{1}{\sqrt{2}}$

$\psi(d,d) = 0 \quad \psi(d,u) = \frac{1}{\sqrt{2}}$

set $b = b'$ in summation

and find entries of ρ

$\rho_{uu} = \psi^*(u,u)\psi(u,u) + \psi^*(u,d)\psi(u,d) = \frac{1}{2}$

$\rho_{ud} = \psi^*(d,u)\psi(u,u) + \psi^*(d,d)\psi(u,d) = 0$

$\rho_{du} = \psi^*(u,u)\psi(d,u) + \psi^*(u,d)\psi(d,d) = 0$

$\rho_{dd} = \psi^*(d,u)\psi(d,u) + \psi^*(d,d)\psi(d,d) = \frac{1}{2}$

$$\rho = \begin{pmatrix} \rho_{uu} & \rho_{ud} \\ \rho_{du} & \rho_{dd} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2} I$$

check $Tr(\rho) = 1$

- Test for entanglement – if entangled the wave function $\psi(a,b)$ and probability function $P(a,b)$ will not factorise ∴ correlated

or assume A is alices observable B is bobs observable
with corresponding expectation values $\langle A \rangle$, $\langle B \rangle$
correlation between them $C(A,B) = \langle AB \rangle - \langle A \rangle \langle B \rangle$
if $C(A,B) \neq 0$ state is entangled if $C(A,B) = 0$ so $\langle AB \rangle = \langle A \rangle \langle B \rangle$
$-1 \leq C(A,B) \leq +1$ than pure state

- for product state (not entangled) the density matrix has only 1 non-zero eigen value equal to exactly 1 ∴ $\sum_a \rho_{a'a} \alpha_a = \lambda \alpha_{a'}$ $\lambda = 1$
the eigen vector of this eigen value of 1 will be the wave function ψ

- for singlet state or any maximally entangled the density matrix is ~~equal to~~ proportional to unit matrix since all eigen values sum up to I (1) eigen value is $\frac{1}{N_A}$ N_A – alices state of states, – number of alices states

$$\rho_{a'a} = \lambda \delta_{a'a}$$
$$= \frac{1}{N_A} \delta_{a'a}$$ kronecker delta so probability of all states is equal

---

- Measurement

take apparatus A with states $|b\rangle$, $|+1\rangle$, $|-1\rangle$ b is blank, +1 is up, -1 is down
spin states $|u\rangle$, $|d\rangle$

both have states since both are quantum states
using tensor product create space of states

$|u,b\rangle$
$|u,+1\rangle$
$|u,-1\rangle$
$|d,b\rangle$
$|d,+1\rangle$
$|d,-1\rangle$

if spin initially in $|u\rangle$ and A in $|b\rangle$
they interact to give A in $|+1\rangle$ so $|u,b\rangle \longrightarrow |u,+1\rangle$

generally initial state of spin $\alpha_u|u\rangle + \alpha_d|d\rangle$
and of system with A $\alpha_u|u,b\rangle + \alpha_d|d,b\rangle$
which gives final state $\alpha_u|u,+1\rangle + \alpha_d|d,-1\rangle$
so $\alpha_u|u,b\rangle + \alpha_d|d,b\rangle \longrightarrow \alpha_u|u,+1\rangle + \alpha_d|d,-1\rangle$

final state is entangled so A entangled with measured system
(must be entangled because from state of A we can tell spin)

expectation value of +1 is still $\alpha_u^* \alpha_u$ untill alice looks at it
but for bob she did not collapse the wave function as he didn't look at system yet she just became entangled with the system if bob looks at her he is also entangled with system but there is still charlie...

- entanglement + locality
  - ↳ John Bell proved QM is non local
  - ↳ locality - impossible to send signal (information) faster than light
  - ↳ even for maximally entangled state Bob can do nothing to immediately affect alices wave function or density matrix

  so one could argue QM is local this is ensured by the operator acting being unitary $U^\dagger = U$

- Bells theorem - any classical simulation ~~which~~ of QM which tries to spatially separate alices and bobs apparatuses and systems must have an instantaneous "cable" connecting the two with central memory to update the state vector (cable only to simulate entanglement)
  - in real world no cable needed ~~since proovem to be~~ but in simulation or boolean computer yes
  - information sent faster than speed of light but not accessed faster so cable even in real life but can not be discovered we know it exists since we need to include it in our simulation.

Summary:

- Product state - no entanglement ∴ excessive locality, seems as classical system
  - each sub system is fully characterized. no correlation between alices and bobs system
  - state vector: $\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle$
  - normalization: $\alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1 \quad \beta_u^*\beta_u + \beta_d^*\beta_d = 1$
  - density matrix has 1 nonzero λ and is = to 1 the eigen vector is the alices wave function same for bob
  - wave function factorised $\psi(a)\,\phi(b)$
  - expectation values $\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 = 1 \quad \langle T_x\rangle^2 + \langle T_y\rangle^2 + \langle T_z\rangle^2 = 1$
  - correlation $\langle\sigma_z T_z\rangle - \langle\sigma_z\rangle\langle T_z\rangle = 0$
- Singlet state - maximum entangled ∴ non locality with all quantum weirdness
  - composite system fully characterised but no information about alices or bobs sub systems
  - state vector: $\frac{1}{\sqrt{2}}\left(|ud\rangle \mp |du\rangle\right)$
  - normalization: $\psi_{uu}^*\psi_{uu} + \psi_{ud}^*\psi_{ud} + \psi_{du}^*\psi_{du} + \psi_{dd}^*\psi_{dd} = 1$
  - density matrix has full composite system $\rho^2 = \rho \quad Tr(\rho^2) = 1$

    alices density matrix is proportional to $\mathbb{I}$ with equal eigen values which all add up to 1 so all measured outcomes are equally likely ← for subsystem of alice $\rho^2 \neq \rho \quad Tr(\rho^2) \neq 1$
  - wave function not factorised - $\psi(a,b)$
  - expectation values $\langle\sigma_x\rangle = \langle\sigma_y\rangle = \langle\sigma_z\rangle = 0 \quad \langle T_x\rangle = \langle T_y\rangle = \langle T_z\rangle = 0$

correlation:
$\langle\sigma_z T_z\rangle - \langle\sigma_z\rangle\langle T_z\rangle = -1$

$\langle T_z\sigma_z\rangle, \langle T_x\sigma_x\rangle, \langle T_y\sigma_y\rangle = -1$

- Almost singlet state - partial entanglement - difficult to tell up - down
  - some information about composite system none about individual systems. In both cases incomplete
  - state vector: $\sqrt{0.6}|ud\rangle - \sqrt{0.4}|du\rangle$
  - normalization: $\psi_{uu}^* \psi_{uu} + \psi_{ud}^* \psi_{ud} + \psi_{du}^* \psi_{du} + \psi_{dd}^* \psi_{dd} = 1$
  - density matrix
    - ↳ full composite system $\rho^2 = \rho$ $Tr(\rho^2) = 1$
    - ↳ alices subsystem $\rho^2 \neq \rho$ $Tr(\rho^2) < 1$
  - wave function not factorised $\psi(a,b)$
  - expectation values $\langle \sigma_z \rangle = 0.2$ $\langle \sigma_x \rangle = \langle \sigma_y \rangle = 0$
    $\langle \tau_z \rangle = -0.2$ $\langle \tau_x \rangle = \langle \tau_y \rangle = 0$
    $\langle \tau_z \sigma_z \rangle = -1$ $\langle \tau_x \sigma_x \rangle = -2\sqrt{0.24}$
  - correlation $\langle \sigma_z \tau_z \rangle - \langle \sigma_z \rangle \langle \tau_z \rangle = -0.96$
    otherwise $-1 < C(a,b) < 1$ $C(a,b) \neq 0$ ↳ partially entangled

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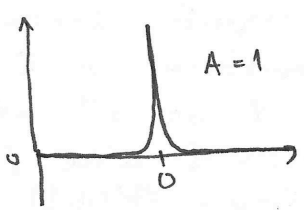
- continuous wave functions
  - ↳ sums => integrals
  - ↳ probability => probability density
  - ↳ kronecker delta => dirac delta
- probability density
  with descrete wave functions $P(x) = \psi^*(x)\psi(x)$
  but with continuous it is near impossible for it to be exactly x
  so we find P of a range of values $P(a,b) = P(a<x<b)$
- so we define a normalized vector by $= \int_a^b \psi^*(x)\psi(x)\,dx$

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)\,dx = 1$$

- Dirac delta function
  - ↳ it is zero for all values $x \neq x'$ but if $x = x'$ it has infinite magnitude. this occours at an infinitesimal range ϵ and the amplitude is $\frac{1}{\epsilon}$ so the area is 1

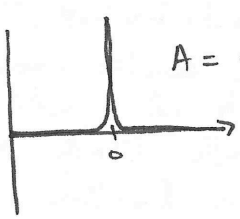
$$\frac{n}{\sqrt{\pi}} e^{-(nx)^2}$$

as $n \to \infty$



$n = 10$

=>



$n = 12$

as n↑ width decreases

- linear operator - "machine" which acts on a function to give another function

↳ linear ∴ - if it acts on ~~the function~~ sum of 2 fuctions it gives sum of individual results
- if it acts on multiple of function it gives multiple of results

eg operator X: $X\psi(x) = x\psi(x)$

D: $D\psi(x) = \frac{d\psi(x)}{dx}$

- for hermitian operators $L^\dagger = L$ it also holds that then between bra and kets are complex conjugates,

$$\langle\Psi|L|\Phi\rangle = (\langle\Phi|L|\Psi\rangle)^*$$

eg.) $\langle\Psi|X|\Phi\rangle = \int \psi^*(x)\, x\, \phi(x)\, dx$ $\langle\Phi|X|\Psi\rangle = \int x\psi(x)\,\phi^*(x)\,dx$

D is not hermitian but anti-hermitian for which

$$D^\dagger = -D$$

we correct this by a constant $-i\hbar$ so we get $\underline{\underline{-i\hbar D}}$

- eigen values and eigenvectors of position

$X|\Psi\rangle = x_0|\Psi\rangle$ so $x\psi(x) = x_0\psi(x)$ we get $\langle x|\Psi\rangle = \psi(x)$

so wave function is projection of state vector $|\Psi\rangle$ onto the eigenvector $|x\rangle$

- eigen values and eigenvectors of momentum

$P = -i\hbar D$ $P\psi(x) = -i\hbar \frac{d\psi(x)}{dx}$ eigen equation: $P|\psi\rangle = p|\psi\rangle$

$$\frac{d\psi(x)}{dx} = \frac{ip}{\hbar}\psi(x)$$

general solution $\psi_p(x) = A e^{\frac{ipx}{\hbar}}$

↳ specific eigenvector of P with eigenvalue p

- inner products of position eigenvectors with momentum eigenvectors in different order are complex conjugates

$\langle x|p\rangle = A e^{\frac{ipx}{\hbar}}$ $\langle x|p\rangle = (\langle p|x\rangle)^*$

$\langle p|x\rangle = A e^{-\frac{ipx}{\hbar}}$

- wave function in terms of momentum - $\tilde{\psi}(p)$

wave function in terms of position - $\psi(x)$

never both - $\psi(x,p)$ ✗ !!

- wavelength of particle with function $e^{\frac{ipx}{\hbar}}$ is

$\lambda = \frac{2\pi\hbar}{p}$ as $p\uparrow$ $\lambda\downarrow$

wavelength since if we add it to position x it doesn't change the wavefunction let $x' = x + \lambda = x + \frac{2\pi\hbar}{p}$

$e^{\frac{ip|x'}{\hbar}} = e^{\frac{ip\,x+\frac{2\pi\hbar}{p}}{\hbar}} = e^{\frac{ipx}{\hbar}} e^{2\pi i} = e^{\frac{ipx}{\hbar}}$ so wt changed

- called a wavefunction since $e^{\frac{ipx}{\hbar}} \Rightarrow e^{ix}$ represents cos/sin waves

---

momentum p $\quad P(p) = |\langle p|\Psi\rangle|^2 \quad \tilde{\Psi}(p) = \langle p|\psi\rangle$

- resolving identity - identity operator ~~can~~ I can be written as

↳ descrete wavefunctions $I = \sum_i |i\rangle\langle i|$

↳ continuous wavefunctions $I = \int dx\,|x\rangle\langle x|$ or $\int dp\,|p\rangle\langle p|$

so using $\tilde{\Psi}(p) = \langle p|\psi\rangle$ and $I = \int dx\,|x\rangle\langle x|$

we get $\tilde{\Psi}(p)\,I = \int dx \underbrace{\langle p|x\rangle\langle x|\psi\rangle}$ so $\tilde{\Psi}(p) = \int dx\,\langle p|x\rangle\langle x|\psi\rangle$

multiplied out $\langle p|\psi\rangle$ sine ~~not~~ $\int$ in dx

---

$x\,p\,\psi(x) = -i\hbar x \frac{d\psi(x)}{dx}$ $\qquad p\,x\,\psi(x) = -i\hbar \frac{d\,\psi(x)\,x}{dx}$ ↘ product rule

commutator $[x,p]\psi(x)$ $\qquad = -i\hbar x \frac{d\,\psi(x)}{dx} - i\hbar\psi(x)$

⇓

$[x,p]\psi(x) = i\hbar\, x\,p\psi(x) - p\,x\,\psi(x)$

$= i\hbar\,\psi(x)$

so $[x,p] = i\hbar$

since if it acts on $\psi(x$ it multiplies it by factor $= i\hbar$

- heisenbergs uncertainty principle

↳ sine x, p don't commute $[x,p] \neq 0$

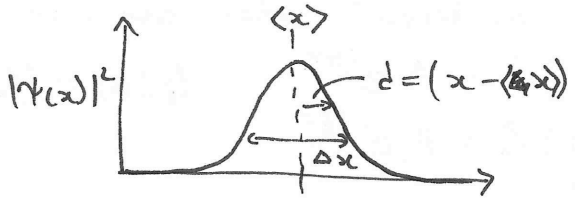
$\Delta x\,\Delta p \geq \frac{1}{2}|\langle\psi|[x,p]|\psi\rangle|$

$\geq \frac{1}{2}|\langle\psi|i\hbar|\psi\rangle|$

$\geq \frac{1}{2}|i\hbar\langle\psi|\psi\rangle|$

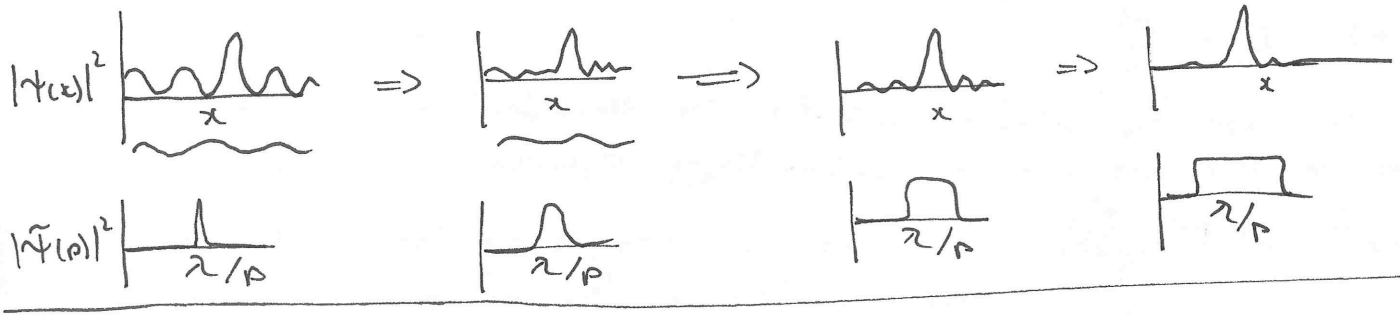
$\geq \frac{\hbar}{2}$ sine $\langle\psi|\psi\rangle = 1$

Δx is measure of how spread out function is in relation to $\langle x\rangle$



- heisenbergs uncertainty principle can be explained with fourier transform

we superimpose waves of slightly different λ to get a single peak therefore know the $\langle x \rangle$ accurately but now we don't know the wavelength



- simple moving particle (non-relativistic)

in this eg simple Hamiltonian operator $H = cp$

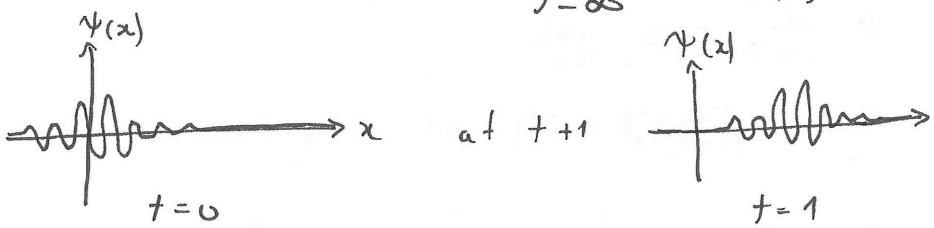
$= c\frac{\hbar}{i}\frac{d}{dx} = -ci\hbar\frac{d}{dx}$

plug into time dependent schrödiger equation $i\hbar\frac{\partial|\Psi\rangle}{\partial t} = H|\Psi\rangle$

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = -ci\hbar\frac{\partial|\psi\rangle}{\partial x} \quad \text{or} \quad i\hbar\frac{\partial\psi(x,t)}{\partial t} = -ci\hbar\frac{\partial\psi(x,t)}{\partial x}$$

which gives: $\frac{\partial\psi(x,t)}{\partial t} = -c\frac{\partial\psi(x,t)}{\partial x}$ solution is $\psi(x,t) = x - ct$

we have to normalize it $\int_{-\infty}^{\infty}\psi^*(x)\,\psi(x)\,dx$ only in terms of space



so wavepacket moved to the right

- classical hamiltonian $H = \frac{1}{2}mv^2 = \frac{p^2}{2m}$ (free ∴ no external energy so potential E term = 0)

to quantize it replace $p \Rightarrow \mathbb{P}$

$\mathbb{H} = \frac{\mathbb{P}^2}{2m}$ $\mathbb{P} = -i\hbar\frac{\partial}{\partial x}$ substituting into $i\hbar\frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle$ gives:

$= \left(-i\hbar\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\frac{1}{2m}$

$= \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

$$i\frac{\partial\psi}{\partial t} = \frac{\hbar}{2m}\frac{\partial^2}{\partial x}$$

this wave function spreads out over time

but λ is constant as long as velocity so $\mathbb{P}$ is constant

- schrödingers tine independant ~~wave fcti~~ equation - $H|\psi\rangle = E|\psi\rangle$

so E is eigenvalue of hamiltonian

substituting we get $-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = E\psi(x)$ assume solution $\psi(x) = e^{\frac{ipx}{\hbar}}$

we get $E = \frac{p^2}{2m}$

- now we can use this solution with the tine dependant one to get the solution to the tine dependant schrödinger equation

$$\psi(x,t) = e^{\frac{i(px - \frac{p^2 t}{2m})}{\hbar}}$$ and any solution is sum or integral of these solutions in tes of p

$$\psi(x,t) = \int \tilde{\psi}(p)\left(e^{\frac{i(px-\frac{p^2t}{2m})}{\hbar}}\right)dp$$ also for wavefunction in tems of momentum $\tilde{\psi}(p,t) = \tilde{\psi}(p)\, e^{\frac{i(px-\frac{p^2t}{2m})}{\hbar}}$

- velocity and momentum

- velocity = tine derivative of average position $v = \frac{d}{dt}\langle\psi|X|\psi\rangle$

in tems of wave functions $v = \frac{d}{dt}\int \psi^*(x,t)\, x\, \psi(x,t)\, dx$

- we know tine derivative of expectation value of observable L is equal to $\frac{i}{\hbar}$ times expectation value of comutator of the hamiltonian H with the observable L $\frac{d}{dt}\langle L\rangle = \frac{i}{\hbar}\langle[H,L]\rangle$

- using hamiltonian from before $v = \frac{i}{2m\hbar}\langle[P^2, X]\rangle$

$[P^2, X] = P[P,X] + [P,X]P$ using $[P,X] = -i\hbar$ which we know

$= -2i\hbar\langle P\rangle$

$v = \frac{2\hbar\langle P\rangle}{2m\hbar} = \frac{\langle P\rangle}{m}$ so $\langle P\rangle = mv$

- Quantization

1.) start with classical system - coordinates x_i and momenta p_i
- hamiltonian equation $H = \ldots$

2.) replace classical phase space with linear vector space
state of states is represented in tems of coordinates by $\psi(x)$
or momentum by $\psi(p)$

3.) replace x_i and p_i in hamiltonian by X_i and P_i which act $X_i\psi = x_i\psi$

4.) hamiltonian is now an operator and can be used in the tine dependant / independant equations $P_i = -\hbar\frac{\partial}{\partial x_i}$

tine dependant - shows development of wave function over tine

tine independant - enables us to calculate eigenvalues + eigenvectors of the hamiltonian corresponding to the Energy

- Forces – force gives rise to potential term in hamiltonian

force can be quantized as potential and becomes operator $\mathbb{V}$ which is added to the hamiltonian to be used $H = T + V$

$\mathbb{V}|\psi\rangle = V(x)\psi(x)$ so operator $\mathbb{V}$ multiplies wave function by a function V

new hamiltonian is $H = \frac{p^2}{2m} + V(x)$ schrodgers: $i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi$

so affects the way ψ chges with time and E $E\psi = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi$

- multiplying by x and by $V(x)$ are both commutative so x and $V(x)$ commute $[x, V(x)] = 0$ so $V(x)$ doesnt affect the before shown commutative relationship and $\langle p\rangle = mv$ still applies

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- classical newtons equations of motion $\frac{dp}{dt} = F$

to try and quantize it calculate $\frac{d}{dt}\langle p\rangle$ using commutation with H

$$\frac{d}{dt}\langle p\rangle = \frac{i}{\hbar}\langle[H, p]\rangle = \frac{i}{2m\hbar}\langle[p^2, p]\rangle + \frac{i}{\hbar}\langle[V, p]\rangle$$

$$= \frac{i}{\hbar}\langle[V, p]\rangle$$

$$= \frac{i}{\hbar}\left\langle i\hbar\frac{dV(x)}{dx}\right\rangle$$

$$= -\left\langle\frac{dV(x)}{dx}\right\rangle$$

commutator commutes with any function of itself so $[p^2, p] = 0$

↓ using relationship

$[V(x), p] = i\hbar\frac{dV(x)}{dx}$

so we get $\frac{d}{dt}\langle p\rangle = -\left\langle\frac{dV(x)}{dx}\right\rangle$

- note: ~~derivative with respect~~ function of average of x ($\langle x\rangle$) is not the same as average of function of x $f(\langle x\rangle) \neq \langle f(x)\rangle$!

- classical limit
- assume incoming wave packet with features of size δx smaller than uncertainty in position so $\delta x < \Delta x$
- if sharp peaks of $V(x)$ due to force exist on scale smaller than incoming wave packet it is scattered so wavefunction brakes up

uncertainty priciple $\Delta x\,\Delta p \geq \frac{\hbar}{2}$

$$\Delta x\, m\,\Delta v \geq \frac{\hbar}{2}$$

$$\Delta x\,\Delta V \geq \frac{\hbar^2}{2m}$$

$$\text{or } \Delta x \geq \frac{\hbar}{2m\Delta V}$$

- gaussian wave packets – $\Delta x\Delta p$ is as small as possible so $\Delta x\Delta p = \hbar$

- Path integrals:

action $A = \int_{t_1}^{t_2} L(x,\dot{x})\,dt$ $\quad L = T - V$ in classical mechanics

here we deal with trajectory between two points

- in quantum mechanics there is no sense in a path between to points so we ask: given particle starts at (x_1, t_1) what is the amplitude (probability) that it will show up at (x_2, t_2)

lets call the amplitude $C(x_1, t_1, x_2 t_2)$ so $C_{1,2}$

initial state $|\psi(t_1)\rangle = |x_1\rangle$ which evolves $|\psi(t_2)\rangle = e^{-iH(t_2-t_1)}|x_1\rangle$

$C_{1,2} = \langle x_2 | e^{-iH(t_2-t_1)} | x_1\rangle$ and define $t = t_2 - t_1$

$= \langle x_2 | e^{-iHt} | x_1\rangle$ using identity operator $I = \int dx\, |x\rangle\langle x|$

$= \int dx\, \langle x_2 | e^{-\frac{iHt}{2}} | x\rangle \langle x | e^{-\frac{iHt}{2}} | x_1\rangle$

power is $\frac{1}{2}$ since we squared it to divide it in the integral

- so for each tiny time interval ϵ
- we include a factor $e^{-i\epsilon H}$
and in integral it gives term $\langle x_i | e^{-i\epsilon H} | x_{i+1}\rangle$
- if we let $U(\epsilon) = e^{-i\epsilon H}$
so we write it as $C_{1,2} = \langle x_2 | U^N | x_1\rangle$

$= \langle x_2 | U U U U \ldots | x_1\rangle$

$= \int dx \langle x_2 | e^{-i\epsilon H} | x\rangle \langle x | e^{-i\epsilon H} | x\rangle \langle x | e^{-i\epsilon H} | x_1\rangle \ldots$

the term N is called the amplitude of the given path so we write $e^{\frac{iA}{\hbar}}$

- A is the action for individual paths $e^{\frac{iA}{\hbar}}$ $\quad C_{1,2} = \int_{\text{paths}} e^{\frac{iA}{\hbar}}$

so amplitude is integral over all possible paths between end points at infinitesimal time intervals $\epsilon \to 0$

↳ important for QED so later.

- classical equation for harmonic oscillator

$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$ $\quad$ canonical momentum $= p = \frac{\partial L}{\partial \dot{x}} = \dot{x}$

so we get $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2$

- quantum harmonic oscillator:

classical $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2$ $\quad X|\psi(x)\rangle = x\psi(x)$

$H = \frac{1}{2}\left(-i\hbar\frac{\partial}{\partial x}\left(-i\hbar\frac{\partial}{\partial x}\psi(x)\right)\right) + \frac{1}{2}\omega^2 x^2\psi(x)$ $\quad P|\psi(x)\rangle = -i\hbar\frac{d}{dx}\psi(x)$

$$H|\psi\rangle = -\frac{\hbar^2}{2}\frac{\partial^2\psi(x)}{\partial x^2} + \frac{1}{2}\omega^2 x^2\psi(x)$$

- substituting into schrodinger equation $; \frac{\partial\psi}{\partial t} = \frac{1}{i\hbar}H\psi$

we get $; \frac{\partial\psi}{\partial t} = -\frac{\hbar}{2}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2i\hbar}\omega^2 x^2\psi$

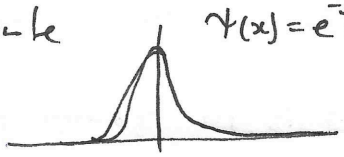
- substituting into $H|\psi_E\rangle = E|\psi_E\rangle$ we can find E levels

$$-\frac{\hbar^2}{2}\frac{\partial^2\psi_E(x)}{\partial x^2} + \frac{1}{2}\omega^2 x^2\psi_E(x) = E\psi_E(x)$$

- most values of E (solutions) grow exponentially as $x \to \infty$ or $-\infty$ so make no physical sense ∴ we create condition:

Is physical solutions to schrödinger equation must be normalisable so filters out majority of E except few special ones

- lowest E state for harmonic oscilator = ground state $\psi(x) = e^{-\frac{\omega x^2}{2\hbar}}$



- ground state: - has no nodes
  - has no zeroes for any potential

ground state wave function $\psi_0(x) = e^{-\frac{\omega}{2\hbar}x^2}$

substituting into schrödinger equation: $\frac{\hbar}{2}\omega e^{-\frac{\omega}{2\hbar}x^2} = Ee^{-\frac{\omega}{2\hbar}x^2}$

ground state function $\psi_0(x)$ is the gaussian. so $E_0 = \frac{\omega\hbar}{2}$

- constructing annihilation / creation operators

hamiltonian in terms of operators $H = \frac{P^2 + \omega^2 X^2}{2}$ using: $a^2 + b^2 = (a+ib)(a-ib)$

but since X, P dont commute we cant just expand it so multiplying out we find we are off by factor of $\frac{\omega\hbar}{2}$

$$= \frac{1}{2}(P + i\omega X)(P - i\omega X)$$

$$\text{actually} = \frac{1}{2}(P + i\omega X)(P - i\omega X) + \frac{\omega\hbar}{2}$$

now define creation operator as - $a^+ = \frac{-i}{\sqrt{2\omega\hbar}}(P + i\omega X)$

annihilation operator as - $a^- = \frac{i}{\sqrt{2\omega\hbar}}(P - i\omega X)$

so hamiltonian becomes: $H = \omega\hbar\left(a^+a^- + \frac{1}{2}\right)$

and their commutator property $[a^-, a^+] = 1$

$$[a^+, a^-] = -1$$

- now define operator $N = a^+ a^-$ - number operator

$H = \omega\hbar\left(N + \frac{1}{2}\right)$

for N we get commutator relationships: $[a^-, a^+] = 1$, $[a^-, N] = a^-$, $[a^+, N] = -a^+$

- create eigenvector/value

$N|n\rangle = n|n\rangle$

and act on it with a^+ $\quad N(a^+|n\rangle) = [a^+N - (a^+N - Na^+)]|n\rangle$

since $|n\rangle$ is eigenvector of N with value n we can replace $(N+1)$ with $(n+1)$ if acting on $|n\rangle$

$= a^+(N+1)|n\rangle$

$= (n+1)(a^+|n\rangle)$

$a^+|n\rangle = |n+1\rangle$

so we found a new eigen vector with eigenvalue increased by 1 so there is ∞ number of eigen vectors above it: $|n+2\rangle, |n+3\rangle \ldots$

annihilation operator does opposite $a^-|n\rangle = |n-1\rangle$

$a^+|n\rangle = |n+1\rangle$ $\quad a^-|n\rangle = |n-1\rangle$

this is the ground state ($|0\rangle$ is not the 0 vector)

but we can't get bellow ground state to -ve E so there must be an eigenvector on which if a^- acts it gives back the same eigenvector so all lower ones are the same $a^-|0\rangle = 0$

- this quantizes E of harmonic oscillator to $E_n = \omega\hbar\left(n + \frac{1}{2}\right)$

$= \omega\hbar\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2} \ldots\right)$

- $a^-|0\rangle = 0$ in terms of ground state $\psi_{(0)}$:

$\frac{i}{\sqrt{2\omega\hbar}}(P - i\omega X)\psi_0(x) = 0$ $\quad$ since $\frac{i}{\sqrt{2\omega\hbar}}$ is const we can divide by it

$(P - i\omega X)\psi_0(x) = 0$

$\frac{d\psi_0}{dx} = -\frac{\omega x}{\hbar}\psi_0$ $\quad$ replacing P with $-i\hbar\frac{d}{dx}$

first order differential eq'n

solution is the gaussian $\psi_0 = e^{-\frac{\omega}{2\hbar}x^2}$

- $\psi_1(x) = (P + i\omega x)\psi_0(x)$

$= \left(-i\hbar\frac{\partial}{\partial x} + i\omega x\right)e^{-\frac{\omega}{2\hbar}x^2}$

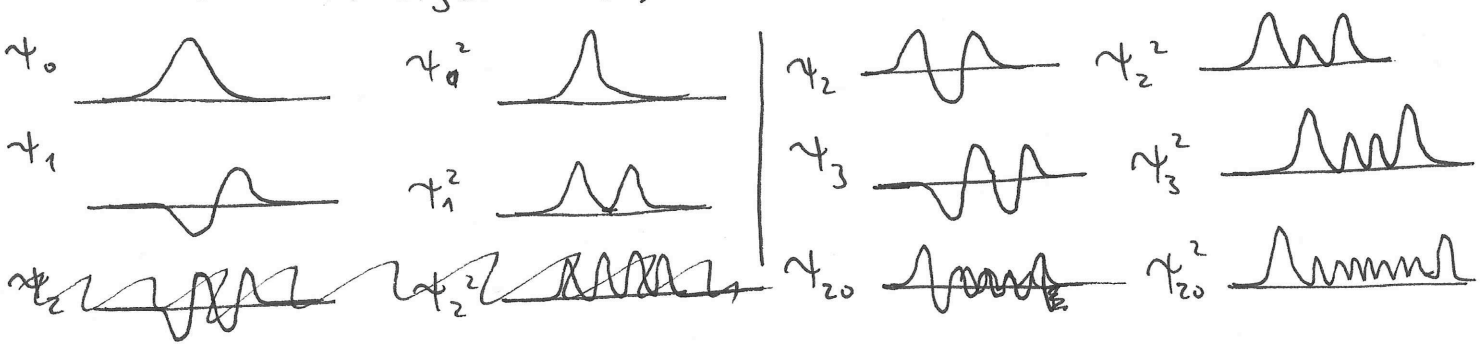
$= i\left(\omega x - \hbar\frac{\partial}{\partial x}\right)e^{-\frac{\omega}{2\hbar}x^2} = i\left(\omega x e^{-\frac{\omega}{2\hbar}x^2} - \hbar\frac{\partial}{\partial x}\left(e^{-\frac{\omega}{2\hbar}x^2}\right)\right)$

$= i\left(\omega x e^{-\frac{\omega}{2\hbar}x^2} + \hbar\, 2x\frac{\omega}{2\hbar}e^{-\frac{\omega}{2\hbar}x^2}\right)$

$= 2i\omega x e^{-\frac{\omega}{2\hbar}x^2}$

(if we go on we get polynomials in terms of x due to rule of $\frac{d}{dx}e^{ux}$) $\rightarrow \psi_1 = 2i\omega x\,\psi_0$

- harmonic oscillator eigenfunctions



- $\omega = \frac{2\pi c}{\lambda}$ but $E \propto f$ so E in length of wave λ has to be $(n+\frac{1}{2})\hbar\omega$ $\frac{1}{2}\hbar\omega$ - zero point E

- ignoring zero point E so $E_\lambda = \frac{2\pi\hbar c}{\lambda} n$ $n \in +\mathbb{Z}$ or 0

so E is quantized in electromagnetic wave $E(\lambda) = \frac{2\pi\hbar c}{\lambda}$

↳ energy of a single photon