Vector Calculus 2022

- 2D scalar field $f = f(x,y)$
  - ↳ partial differentiation $\frac{\partial f}{\partial x} = \left(\frac{\partial f(x,y)}{\partial x}\right)_y = \lim_{\delta x \to 0} \frac{f(x+\delta x,y) - f}{\delta x}$
  - ↳ total differential $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ (tangent plane)

- Clairaut's theorem
  - ↳ if 1st and 2nd derivatives of $f(x,y)$ are defined, continuu

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

- 2D vector field $\vec{A}(x,y) = A_x(x,y)\hat{i} + A_y(x,y)\hat{j}$
  - ↳ partial differentiation $\frac{\partial \vec{A}}{\partial x} = \lim_{\delta x \to 0} \frac{\vec{A}(x+\delta x,y) - \vec{A}(x,y)}{\delta x}$

$$\frac{\partial \vec{A}}{\partial x} = \frac{\partial A_x}{\partial x}\hat{i} + \frac{\partial A_y}{\partial x}\hat{j} \qquad \frac{\partial \vec{A}}{\partial y} = \frac{\partial A_x}{\partial y}\hat{i} + \frac{\partial A_y}{\partial y}\hat{j}$$

  - ↳ differential $d\vec{A} - \frac{\partial \vec{A}}{\partial x}dx + \frac{\partial \vec{A}}{\partial y}dy$

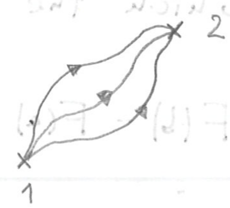
- exact differential
  - ↳ differential is exact if: $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

$$P(x,y) = \frac{\partial f}{\partial x} \quad Q(x,y) = \frac{\partial f}{\partial y} \quad \text{and} \quad \underline{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{\partial f}{\partial x \partial y}}$$

  - ↳ exact differential is a total differential of some parent function f

$$f = \int \frac{\partial f}{\partial x}dx + C(y) = \int \frac{\partial f}{\partial y}dy + C(x)$$

  - ↳ path integral of exact differential between 2 points is independant of the path



- chain rule with partial derivatives

↳ when $y = y(x)$ but we cant substitute for y

for example $y + y^2 = x^3 + x^5$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\frac{dy}{dx}dx \Longrightarrow \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

↳ when x, y are defined parametrically $x = x(t)$ $y = y(t)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt}dt + \frac{\partial f}{\partial y}\frac{dy}{dt}dt \Longrightarrow \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

↳ change of variables $x = x(u,v)$ $y = y(u,v)$

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \qquad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dV$$

to get $\frac{\partial f}{\partial u}$ keep v constant so $dv = 0$

$$dx = \frac{\partial x}{\partial u}du \quad dy = \frac{\partial y}{\partial u}du \qquad df = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u}du + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}du$$

to get $\frac{\partial f}{\partial v}$ keep u constant so $du = 0$

$$dx = \frac{\partial x}{\partial v}dv \quad dy = \frac{\partial y}{\partial v}dv \qquad df = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v}dv + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}dV$$

- fundemental theorem of calculus

↳ first integral theorem

↳ rate of change of area is equal to the function under which the area is bounded

$$F(b) - F(a) = \int_a^b f(x)\,dx \qquad \text{where } f(x) = \frac{dF}{dx}$$

- 1D integration
  - integral of $f(x)$ from a to b is the limit as $n \to \infty$ of the Riemann sum

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)(\xi_i - \xi_{i-1})$$

x_i is a point between ξ_i and ξ_{i-1}

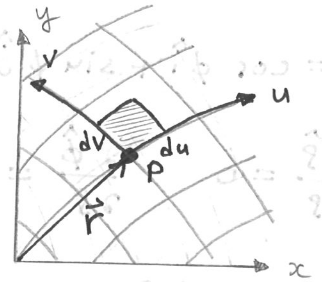
  - think of integral as a weighted sum where "weight" is the height of the bars

- 2D integration
  - integral of $f(x,y)$ over region R is the limit as $n \to \infty$ of the Riemann sum also $\delta A_i \to 0$

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\,\delta A_i$$

- convention
  - evaluate integrals from the inside out
  - order of integrals determines the limits used

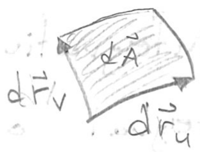
- to change variables we have to change:
  1.) integrand 2.) limits 3.) differential



- the Jacobian
  - draw a vector $\vec{r}$ from origin to point
  - $\vec{r}$ forms a vector field $\vec{r} = x\hat{i} + y\hat{j}$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u}du + \frac{\partial \vec{r}}{\partial v}dv = d\vec{r}_u + d\vec{r}_v \qquad d\vec{r}_u = \frac{\partial \vec{r}}{\partial u}du \qquad d\vec{r}_v = \frac{\partial \vec{r}}{\partial v}dv$$

↳ now the area $d\vec{A}$ is found as the cross-product of $d\vec{r}_u$ and $d\vec{r}_v$



$$d\vec{r}_u = \frac{\partial \vec{r}}{\partial u}\,du = \left(\frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j}\right)du$$

$$d\vec{r}_v = \frac{\partial \vec{r}}{\partial v}\,dv = \left(\frac{\partial x}{\partial v}\hat{i} + \frac{\partial y}{\partial v}\hat{j}\right)dv$$

$$d\vec{A} = d\vec{r}_u \times d\vec{r}_v = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \hat{k}\,du\,dv$$

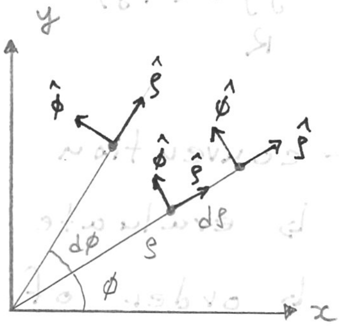
$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \hat{k} = \frac{\partial(x,y)}{\partial(u,v)}$$

↳ jacobian is the ratio of unit areas in the two systems

- transforming a 2D integral $\iint\limits_{R_{xy}} f(x,y)\,dx\,dy \rightarrow \iint\limits_{R_{uv}} f(x(u,v),y(u,v))\,|J|\,du\,dv$

- Jacobian for plane-polar coordinates $|J| = \rho$

- differentiating vectors in plane polars



↳ moving radially out by $d\rho$ doesn't change unit vectors

↳ moving angularly by $d\phi$ changes both $\hat{\phi}$ and $\hat{\rho}$

$\hat{\phi} = \hat{\phi}(\phi) \quad \hat{\rho} = \hat{\rho}(\phi)$

$\hat{\rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j} \qquad \hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

$$\frac{\partial \hat{\rho}}{\partial \rho} = 0 \qquad \frac{\partial \hat{\phi}}{\partial \rho} = 0 \qquad \frac{\partial \hat{\rho}}{\partial \phi} = \hat{\phi} \qquad \frac{\partial \hat{\phi}}{\partial \phi} = -\hat{\rho}$$

↳ to differentiate vector field $\vec{A}$ use product rule

$$\vec{A} = A_\rho(\rho,\phi)\,\hat{\rho}(\phi) + A_\phi(\rho,\phi)\,\hat{\phi}(\phi)$$

$$\frac{\partial \vec{A}}{\partial \phi} = \frac{\partial}{\partial \phi}(A_\rho\hat{\rho}) + \frac{\partial}{\partial \phi}(A_\phi\hat{\phi}) = \frac{\partial A_\rho}{\partial \phi}\hat{\rho} + A_\rho\hat{\phi} + \frac{\partial A_\phi}{\partial \phi}\hat{\phi} - A_\phi\hat{\rho}$$

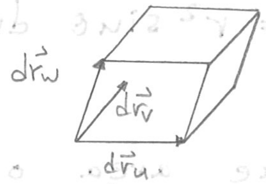
- integral in 3D
  ↳ summing of infinitesimal tetrahedrons (in Cartesian axes)

- Jacobian in 3D

$x = x(u,v,w) \quad y = y(u,v,w) \quad z = z(u,v,w)$

differential of $\vec{r}$: $d\vec{r} = \frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv + \frac{\partial \vec{r}}{\partial w} dw = d\vec{r}_u + d\vec{r}_v + d\vec{r}_w$

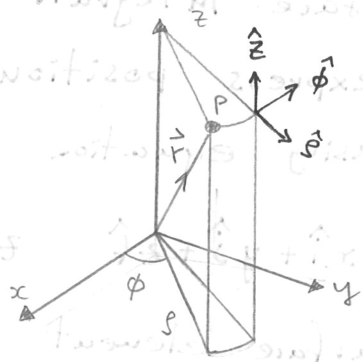
$d\vec{r}_u = \frac{\partial \vec{r}}{\partial u} du = \left(\frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j} + \frac{\partial z}{\partial u}\hat{k}\right) du$



find volume of parallelipiped $d\vec{r}_u, d\vec{r}_v, d\vec{r}_w$ $\quad V = \vec{a}\cdot\vec{b}\times\vec{c}$

$$dV = \left(\frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j} + \frac{\partial z}{\partial u}\hat{k}\right)\cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} \qquad J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- cylindrical polar coordinates
  ↳ unit vectors are independent of ρ and z only depend on ϕ
  ↳ orthogonal system just like plane polars



$x = \rho\cos\phi \qquad \rho = \sqrt{x^2+y^2}$

$y = \rho\sin\phi \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right)$

$z = z \qquad z = z$

- Jacobian for cylindrical polar coordinates $|J| = \rho$

- Spherical polar coordinates
  ↳ orthogonal system

$\hat{r}(\theta,\phi) = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta$

$\hat{\theta}(\theta,\phi) = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta$

$\hat{\phi}(\phi) = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

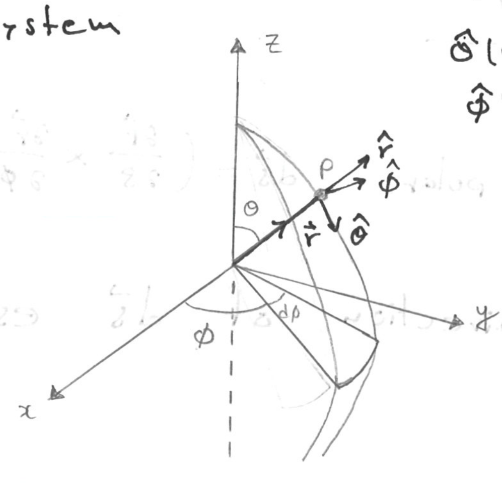
$x = r\sin\theta\cos\phi$

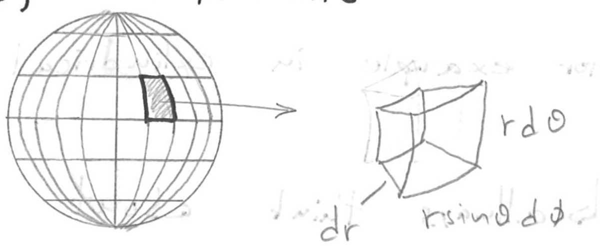
$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$0 < \theta < \pi$

$0 < \phi < 2\pi$





- Jacobian for spherical polar coordinates

position vector: $\vec{r} = r\hat{r}(\theta, \phi)$

differential: $d\vec{r} = \frac{\partial \vec{r}}{\partial r} dr + \frac{\partial \vec{r}}{\partial \theta} d\theta + \frac{\partial \vec{r}}{\partial \phi} d\phi$

$\frac{\partial \vec{r}}{\partial \theta} = \hat{\theta}$

$\frac{\partial \vec{r}}{\partial \phi} = \sin\theta \, \hat{\phi}$

$\frac{\partial \vec{r}}{\partial r} = 0$

$d\vec{r}_r = dr\,\hat{r} \quad d\vec{r}_\theta = r\,d\theta\,\hat{\theta} \quad d\vec{r}_\phi = r\sin\theta\,d\phi\,\hat{\phi}$

$dV = r^2 \sin\theta \, dr\, d\theta\, d\phi \qquad |J| = r^2 \sin\theta$

surface area of revolution (about y axis) $\quad A = 2\pi \int_{t_1}^{t_2} x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

- volume of revolution (about y axis) $\quad V = \pi \int_{y_1}^{y_2} x^2\, dy$

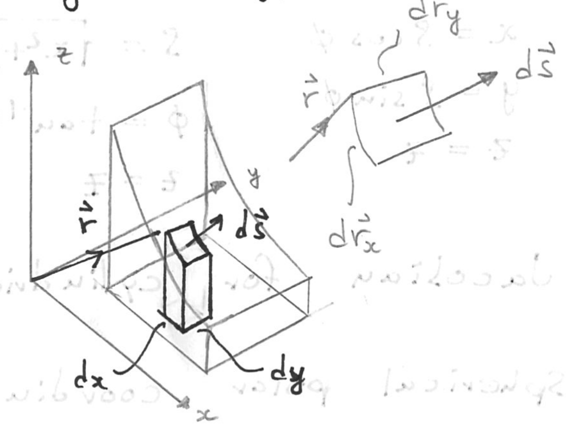
- Surface integrals
  - express position vector $\vec{r}$ of a point on surface using equation of surface so $N-1$ variables (N - no of dim)

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad z = f(x,y) \Rightarrow \vec{r} = x\hat{i} + y\hat{j} + f(x,y)\hat{k}$

- surface element

$d\vec{r} = \frac{\partial \vec{r}}{\partial x} dx + \frac{\partial \vec{r}}{\partial y} dy = d\vec{r}_x + d\vec{r}_y$

$d\vec{S} = d\vec{r}_x \times d\vec{r}_y$



generally: $d\vec{S} = \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) du\, dv$

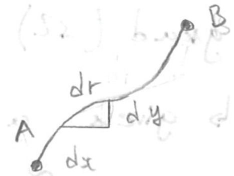
for example in cylindrical polars: $d\vec{S} = \left(\frac{\partial \vec{r}}{\partial \rho} \times \frac{\partial \vec{r}}{\partial \phi}\right) d\rho\, d\phi$

- allways think about direction of $d\vec{S}$ especially when integrating flux

- flux ϕ of a vector field $\vec{F}$ through S $\quad \phi = \iint_S \vec{F} \cdot d\vec{S}$

- Line integrals in 2D
  ↳ curve $y = y(x)$ and vector field $\vec{F}(x,y) = F_x(x,y)\hat{i} + F_y(x,y)\hat{j}$

$$\int_A^B \vec{F} \cdot d\vec{r} = \int_{x_1}^{x_2} F_x(x, y(x))\, dx + \int_{y_1}^{y_2} F_y(x(y), y)\, dy$$



↳ we can rewrite $y(x)$ in terms of x and $x(y)$ in terms of y

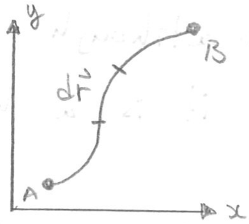
↳ alternatively use 1D Jacobian to transform integrals

$$\int_{x_1}^{x_2} F_x(x, y(x))\, dx \Longrightarrow \int_{y_1}^{y_2} F_x(x(y), y)\, \overbrace{\frac{d}{dy}(x(y))\, dy}^{dx}$$

- Line integrals in 3D
  ↳ similar to 2D since point on line is allways defined by one variable

- generally line integral depends on the path taken

- line integral of an exact differential



↳ we are summing the changes of $\Omega(x,y)$ which are $d\Omega$ along a path from A to B
so overall integral is change of Ω from A to B

↳ line integral of an exact differential is independent of path

↳ the parent function represents the potential of a conservative force field

↳ the closed path integral of exact differential is 0

$$\oint_C d\Omega = 0$$

$$\underline{\vec{F} \cdot d\vec{r} = d\Omega}$$

$$\int_A^B d\Omega(x,y) = \Omega(B) - \Omega(A)$$

- del (nabla) operator ∇

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\nabla\Omega = \frac{\partial\Omega}{\partial x}\hat{i} + \frac{\partial\Omega}{\partial y}\hat{j} + \frac{\partial\Omega}{\partial z}\hat{k}$$

- grad $(\Omega) = \nabla\Omega$

↳ given a scalar field Ω returns a vector field

↳ the vector field is conservative since Ω is the parent function

$$\oint \nabla\Omega \cdot d\vec{r} = 0$$

- directional derivative

$$\frac{d\Omega}{ds} = \nabla\Omega \cdot \hat{a}$$

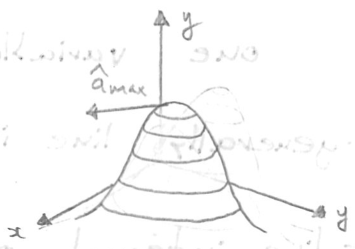
$\hat{a}$ - direction

$$d\Omega = \frac{\partial\Omega}{\partial x}dx + \frac{\partial\Omega}{\partial y}dy + \frac{\partial\Omega}{\partial z}dz = \nabla\Omega \cdot d\vec{r}$$

moving distance ds in direction $\hat{a}$ $\quad d\vec{r} = ds\,\hat{a}$

↳ in the direction where gradient is greatest $\hat{a}_{max}$

$\hat{a}$ is along $\nabla\Omega$ so $\hat{a}_{max} = \frac{\nabla\Omega}{|\nabla\Omega|}$



↳ although $\hat{a}_{max}$ is direction of greatest gradient it is a horizontal vector in x,y plane

- grad in cylindrical polar coordinates

$$\nabla\Omega = \frac{\partial\Omega}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\Omega}{\partial\phi}\hat{\phi} + \frac{\partial\Omega}{\partial z}\hat{k}$$

$$\nabla = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + k\frac{\partial}{\partial z}$$

- grad in spherical polar coordinates

$$\nabla\Omega = \frac{\partial\Omega}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Omega}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\Omega}{\partial\phi}\hat{\phi}$$

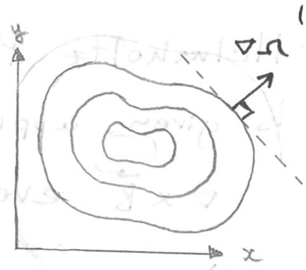
$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

- $d\Omega = \nabla\Omega \cdot d\vec{r}$ has to hold for any coordinate system

then just find $\nabla\Omega$ which satisfies it.

- normal to a surface

↳ in 2D:



$\hat{a}_{max}$ is tangent to a contour so $\frac{d\Omega}{ds} = 0$ since Ω is constant. $\frac{d\Omega}{ds} = \nabla\Omega \cdot \hat{a}$ and $\hat{a}_{max}$ is along contour so $\nabla\Omega$ is $\perp$ to contour

↳ in 3D

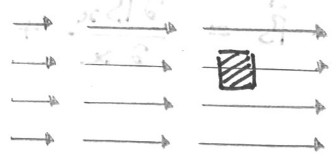
we have a surface of constant Ω so $\nabla\Omega$ is $\perp$ to it and normal to the surface

↳ for a surface defined as $z = f(x,y)$

normal: $\hat{n} = \frac{\nabla\Omega}{|\nabla\Omega|}$ where $\Omega = f(x,y) - z$

- divergence

↳ total amount of flux coming out of an infinitesimal volume element in a vector field

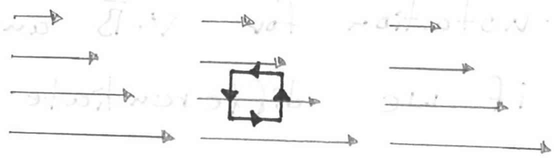


$$\nabla \cdot \vec{B} = \lim_{V \to 0} \left(\underbrace{\frac{1}{V}}_{\text{divide by vol to get flux density}} \underbrace{\oiint_S}_{\text{closed surface}} \underbrace{\vec{B} \cdot d\vec{s}}_{\text{flux out of a region}} \right)$$

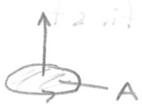
- curl

↳ circulation of a vector field in an infinitesimal area

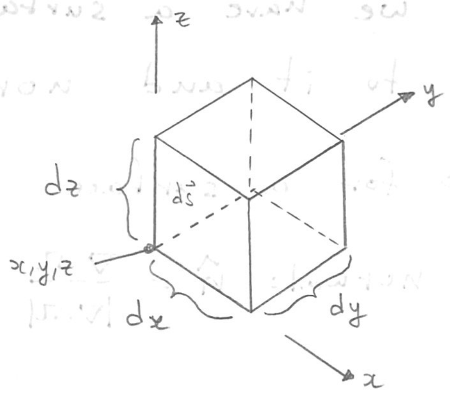
↳ think of it as flow of a river. if the flow velocity at the centre is smaller than at the edge a leaf will rotate



$$\nabla \times \vec{B} = \lim_{A \to 0} \left(\underbrace{\frac{1}{A}}_{\text{divide by area to get circulation surface density}} \underbrace{\oint \vec{B} \cdot d\vec{r}}_{\text{circulation}} \right)$$



- Helmholtz theorem
  - given appropreate boundary conditions knowing $\nabla \cdot \vec{B}$ and $\nabla \times \vec{B}$ everywhere is sufficient to know the field $\vec{B}$
  - $\vec{B}$ can be decomposed to irrotational $(\nabla \times \vec{B} = 0)$ and solenoidal $(\nabla \cdot \vec{B} = 0)$ vector fields
  - called fundemental theorem of vector calculus

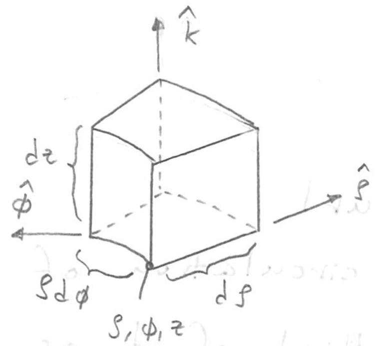


- Divergence in cartasean coordinates
  - use total differential to get 2D "tangent plane" approx for vector field $\vec{B}$ so avg value over each face is at centre
  - multiply by surface of face e.g. $d\vec{s} = -\hat{i}\, dy\, dz$
  - then apply limit $dx, dy, dz \to 0$

$$\nabla \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

dot product definition only works in Cartaseau coordinates !

- Divergence in cylindrical polar coordinates



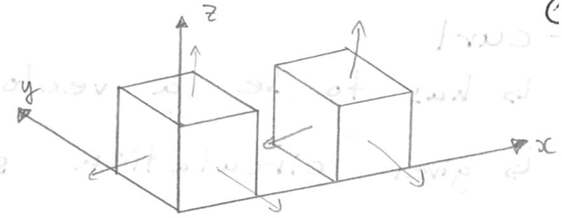
$$\nabla \cdot \vec{B} = \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} \right)$$

- Divergence in spherical polar coordinates

$$\nabla \cdot \vec{B} = \frac{1}{\rho^2}\frac{\partial}{\partial r}(r^2 B_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, B_\theta) + \frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial \phi}$$

- notation for $\nabla \cdot \vec{B}$ and $\nabla \times \vec{B}$ works only if we differentiate first then take the dot / cross product

- Divergence theorem (Gauss theorem)



↳ consider 2 infinitesimal volume elements and find the flux through each face

↳ push the boxes together and sum the flux $\vec{B}\cdot d\vec{s}$ together the flux on touching sides cancels out so the overall flux only depends on the surface flux

$$\oiint \vec{B}\cdot d\vec{s} = \iiint \nabla\cdot\vec{B}\, dV$$

flux over a closed surface — sum of divergence at all points within a volume

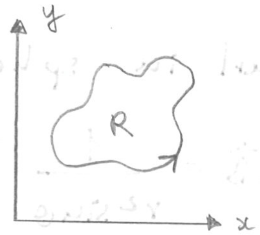
convert surface integral to volume integral and vice versa

- Laplacian $\quad \nabla\cdot(\nabla\Omega) = \nabla^2\Omega = \frac{\partial^2\Omega}{\partial x^2} + \frac{\partial^2\Omega}{\partial y^2} + \frac{\partial^2\Omega}{\partial z^2}$

- Laplace equation $\quad \nabla^2 u = 0 \quad \Rightarrow$ wave equ $\quad \nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

- Greens theorem in the plane

$$\oint P(x,y)\,dx + Q(x,y)\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\, dy$$



where P, Q are any continuous functions

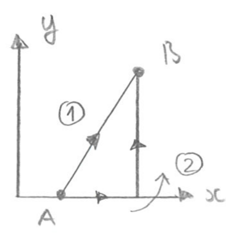
↳ if $P\,dx + Q\,dy$ is an exact differential so $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$$\oint P\,dx + Q\,dy = \iint_R \underbrace{\cancel{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}}_{0}\, dx\, dy = 0$$

↳ Stoke's theorem in 2D $\quad \oint \vec{F}\cdot d\vec{r} = \oint F_x\,dx + \oint F_y\,dy$

so we get $\quad \oint \vec{F}\cdot d\vec{r} = \iint \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\, dx\, dy$

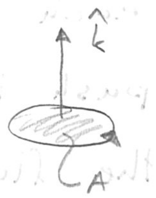
↳ change line integral ⇔ 2D integral



find paths ① and ② and add to get loop integral

- curl
  - ↳ has to be a vector since there is axis of rotation
  - ↳ gives circulation surface density of a vector field

- Curl in cartasean coordinates



  - ↳ derive $\hat{k}$ component of curl for loop of finite size and take limit as $A = \iint dx\, dy \to 0$

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

in 2D $\nabla \times \vec{B} = \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \hat{k}$

- Curl in cylindrical polar coordinates

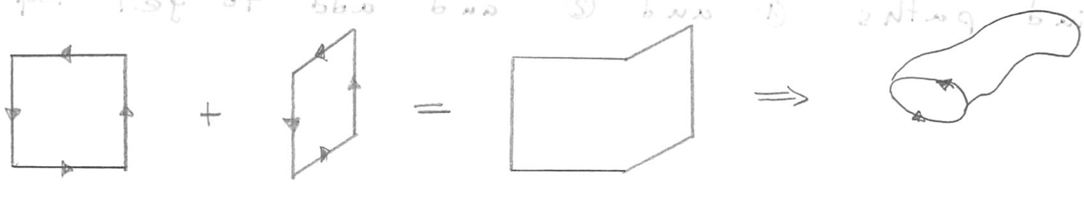
$$\nabla \times \vec{B} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{k} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ B_\rho & \rho B_\phi & B_z \end{vmatrix}$$

- Curl in spherical polar coordinates

$$\nabla \times \vec{B} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\, \hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ B_r & r B_\theta & r\sin\theta\, B_\phi \end{vmatrix}$$

- Curl of a conservative field
  - ↳ for a conservative field $\oint_C \vec{F} \cdot d\vec{r} = 0$ so $\nabla \times \vec{F} = 0$
  - ↳ conservative field also called irrotational

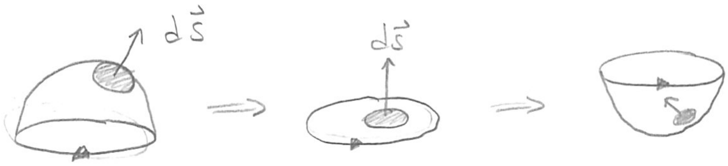
- Stokes theorem
  - ↳ consider 2 adjacent infinitesimal loops in different planes and join them then repeat to build up a surface around a loop



- Stokes theorem (continued)

↳ $\nabla \times \vec{B} \cdot d\vec{s} = \oint \vec{B} \cdot d\vec{r}$ for an infinitesimal loop

$$\oint_C \vec{B} \cdot d\vec{r} = \iint_S \nabla \times \vec{B} \cdot d\vec{s}$$



↳ use RHR by collapsing the surface into the loop and then applying right hand rule

↳ two surfaces attached to same loop have the same surface integral $\iint_S \nabla \times \vec{B} \cdot d\vec{s}$

↳ Stokes theorem can be used for:

- change closed line integral ⟷ surface integral
- change surface integral ⟷ any surface integral provided loop stays constant

- Vector identities

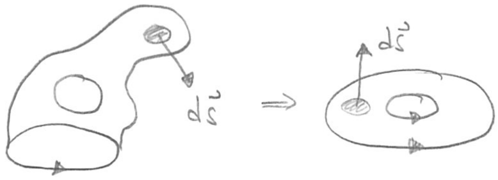
↳ $\nabla \times (\nabla \Omega) = 0$ curl of conservative vector field is 0

↳ $\nabla \cdot (\nabla \times \vec{V}) = 0$ divergence of a curl is 0 (solenoidal)

$\vec{V}$ is called vector potential

↳ $\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B}$

- Hole in surface for stokes theorem



$$\iint_{\text{surface only}} \nabla \times \vec{B} \cdot d\vec{s} = \oint_{\text{whole loop}} \vec{B} \cdot d\vec{r} - \oint_{\text{hole}} \vec{B} \cdot d\vec{r}$$