Fourier Analysis 2022

- complex functions $\quad f(z) = u(z) + iv(z) \quad f(z),\ z \in \mathbb{C} \quad u(z), v(z) \in \mathbb{R}$

$$u(z) = \frac{f(z) + f^*(z)}{2} \qquad v(z) = \frac{f(z) - f^*(z)}{2i}$$

- parity $\quad$ odd: $f(x) = -f(-x)$ $\quad$ even: $f(x) = f(-x)$
  - ↳ parity of a function depends on the point we define it abou
  - ↳ any function can be written as sum of even + odd

$$f(x) = e(x) + o(x) \qquad e(x) = \frac{f(x) + f(-x)}{2} \qquad o(x) = \frac{f(x) - f(-x)}{2}$$

  - ↳ properties: $\underset{-}{o} \times \underset{-}{o} = \underset{+}{e} \qquad \underset{+}{e} \times \underset{+}{e} = \underset{+}{e} \qquad \underset{+}{e} \times \underset{-}{o} = \underset{-}{o}$

$$\int_{-a}^{a} e(x)\,dx = 2\int_{-a}^{a} e(x)\,dx \qquad \int_{-a}^{a} o(x)\,dx = 0$$

- orthogonal vector set: $\{\vec{V}_n\} \Rightarrow \vec{V}_n \cdot \vec{V}_m = 0$ if $n \neq m$
  - ↳ no of vectors is the no of dimensions of the vector space

- orthonormal vector set: $\{\vec{V}_n\} \Rightarrow \vec{V}_n \cdot \vec{V}_m = \delta_{nm} \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$

- given a complete orthonormal set any vector $\vec{A}$ can be writte as $\vec{A} = \sum_{n=1}^{N} a_n \vec{V}_n$ where $a_n = \vec{A} \cdot \vec{V}_n$

- for complex orthonormal vector set $\vec{V}_n \cdot \vec{V}_m^* = \delta_{nm}$

- inner product of functions

$$\langle f | g \rangle = \int_a^b f(x)\, g^*(x)\, dx \quad \text{where } f(x), g(x) \in \mathbb{C}$$

orthonormal functions set $\{f_n\}$

$\langle f_n | f_m \rangle = \delta_{nm} \Rightarrow \int_a^b f_n(x)\, f_m^*(x)\, dx = \delta_{nm}$

↳ if this set is complete we have orthonormal basis functions

to decompose any function $f(x)$ into basis set $\{g_n\}$

$f(x) = \sum_{n \in \mathbb{Z}} a_n g_n(x) \qquad a_n = \langle f | g_n \rangle = \int_a^b f(x)\, g_n^*(x)\, dx$

· comments: - view functions as generalisation of vectors

- view inner product as a mapping of 2 vectors onto scalar

properties of inner product

↳ additivity : $\langle x+y | z \rangle = \langle x | z \rangle + \langle y | z \rangle$

↳ linearity : $\langle ax | y \rangle = a \langle x | y \rangle$

↳ conjugate symmetry : $\langle x | y \rangle = \langle y | x \rangle^*$

↳ positive definite : $\langle x | x \rangle \geq 0$ (if $\langle x | x \rangle = 0 \Rightarrow x = 0$)

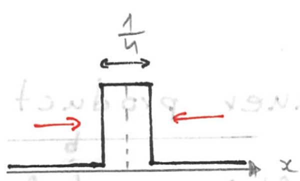
- Dirac delta function

↳ defined as : $\delta(x) = 0 \quad x \neq 0$ and $\int_a^b f(x)\, \delta(x)\, dx = f(0)$

↳ shifting property: $f(x) = \int_{-\infty}^{\infty} f(t)\, \delta(t-x)\, dt$

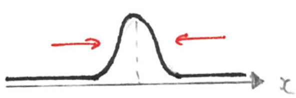
↳ limiting sequences give $\delta(x)$ as $\lim_{n \to \infty}$

$\delta_n(x) \begin{cases} n & \text{if } |x| \leq \frac{1}{2n} \\ 0 & \text{if } |x| > \frac{1}{2n} \end{cases}$



$\delta_n(x) = \frac{1}{2\pi} \int_{-n}^{n} e^{ixt}\, dt \quad (\mathrm{sinc}(nx))$

$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2} \quad (\text{gaussian})$



- use the orthonormal basis functions $\{g_n(x)\}$ $\quad g_n(x) = \frac{1}{\sqrt{2\pi}} e^{in\theta}$

$n \in \mathbb{Z}$

- square integrable functions

  ↳ functions for which $\langle f|f\rangle$ is finite $\quad \langle f|f\rangle = \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta < \infty$

- Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

monopole term $\frac{a_0}{2}$ is the average $\quad \frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$

↳ for functions with different period $\quad 2\pi \rightarrow L \quad \pi \rightarrow \frac{L}{2} \quad -\pi \rightarrow -\frac{L}{2}$

- exponential Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad c_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-inx}\, dx$$

- reality condition

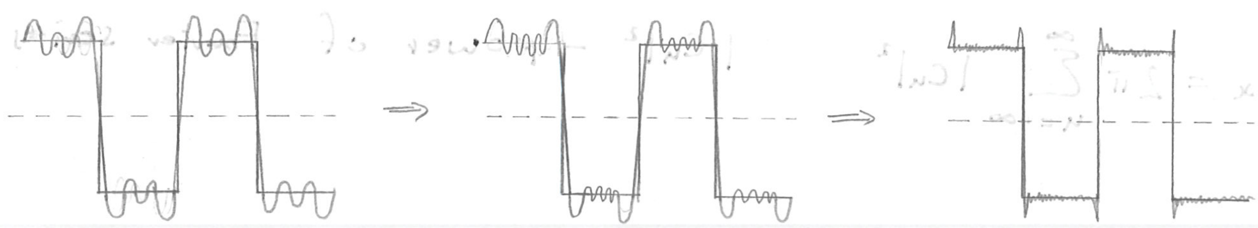
  ↳ if $f(x) \in \mathbb{R}$ $\quad c_n \in \mathbb{C}$ we are calculating some redundant information in $\mathrm{Im}(c_n)$

  ↳ $c_n^* = c_{-n}$ all $-$ve n coefficients are conjugates of +ve n

- Gibbs phenomenon

  ↳ Fourier sums overshoot at jump disontinuities and the overshoot doesn't disappear as $n \rightarrow \infty$

  ↳ solved by Fejér summation or Riesz summation



- Dirichlet conditions

↳ conditions for $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ to hold

↳ if:

1.) $f(x)$ must be periodic over a 2π interval

2.) $f(x)$ must be single valued

3.) $f(x)$ has a finite number of min/max over the interval

has finite freq

4.) $f(x)$ has a finite number of discontinuities (Gibbs)

5.) interval of $|f(x)|$ over interval is finite

↳ then:

1.) series converges to $f(x)$ at all points where $f(x)$ is continuous

2.) series converges to midpoint of $f(x)$ at discontinuities

- Fourier series for interval $-l \to l$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{l} x\right) + b_n \sin\left(\frac{n\pi}{l} x\right) \right)$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi}{l} x\right) dx \qquad b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi}{l} x\right) dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\frac{\pi}{l}x} \qquad c_n = \frac{1}{2l} \int_{-l}^{l} f(x)\, e^{-in\frac{\pi}{l}x} dx$$

- power spectrum

↳ rewrite Fourier series as $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \underbrace{\alpha_n}_{\text{amplitude}} \cos(nx - \underbrace{\theta_n}_{\text{phase}})$

↳ power $= |a_n|^2$ phase $= \theta_n$

$$\alpha_n^2 = a_n^2 + b_n^2 \qquad \tan(\theta_n) = \frac{b_n}{a_n}$$

- Parsevals identity

$|f(x)|^2$ - power of $f(x)$

$|c_n|^2$ - power of Fourier series

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2$$

- Parsevals inequality

$$\sum_{n=-N}^{N} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

↳ to determine how close truncated series is determine

$$1 - \frac{\sum_{n=-N}^{N} |c_n|^2}{\sum_{n=-\infty}^{\infty} |c_n|^2}$$

at $N = \infty$ gives 0

at $N = 0$ gives 1

- Fourier transformation

↳ complete derivation is beyond the scope of these notes

$$\underbrace{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{-ikx} \, dk}_{\mathcal{F}^{-1}(g(t)) \text{ - reverse}} \qquad \underbrace{g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} \, dx}_{\mathcal{F}(f(x)) \text{ - forwards}}$$

↳ x - spatial coordinate k - reciprocal variable $x \longleftrightarrow k$

$t \longleftrightarrow \omega$

- F.T. of a gaussian

↳ normalised gaussian (area = 1) $f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{t^2}{\sigma^2}}$

$$g(\omega) = \mathcal{F}(f(t)) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma^2\omega^2}$$

so reciprocal standard deviation and scaled up by σ

↳ has reciprocal width $\rho = \frac{1}{\sigma}$

- F. T. of a delta function

↳ consider the gaussian limiting function

↳ since delta function is narrow $\sigma = 0$ so $\rho = \frac{1}{\sigma} \to \infty$

and gives a constant value $\frac{1}{\sqrt{2\pi}}$

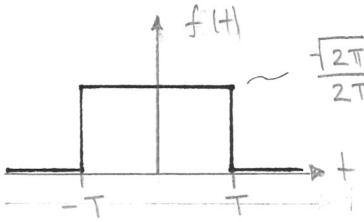
$$g(\omega) = \mathcal{F}(\delta(t)) = \frac{1}{\sqrt{2\pi}}$$

- F.T. of a top-hat function

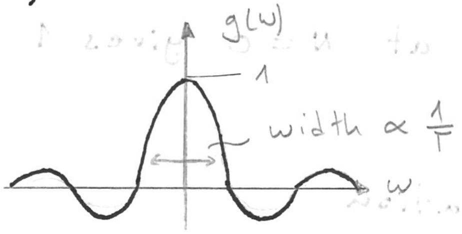
↳ describes what happens when we have finite time interval of measurements

normalise amplitude to $\frac{\sqrt{2\pi}}{2T}$ $\quad f(t) = \begin{cases} \frac{\sqrt{2\pi}}{T} & \text{for } -T \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$

$$g(\omega) = \mathcal{F}(f(t)) = \text{sinc}(\omega T)$$



$\Longrightarrow$



↳ as $T \to \infty$ we get delta function in frequency domain
as $T \to 0$ we get constant value in frequency domain

- properties of Fourier transforms

↳ sign reversal: $\mathcal{F}[f(-t)] = g(-\omega)$

↳ conjugation: $\mathcal{F}[f^*(t)] = g^*(-\omega)$

↳ linearity: $\mathcal{F}[\alpha f_1(t) + \beta f_2(t)] = \alpha \mathcal{F}[f_1(t)] + \beta \mathcal{F}[f_2(t)]$

↳ translation: $\mathcal{F}[f(t-t_0)] = e^{i\omega t_0} g(\omega)$

↳ derivative: $\mathcal{F}[f'(t)] = -i\omega g(\omega)$

↳ scaling: $\mathcal{F}[f(\alpha t)] = \frac{1}{|\alpha|} g\left(\frac{\omega}{\alpha}\right)$

↳ Parsevals identity still holds

power conserved

$$\int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt = \int_{-\infty}^{\infty} F(\omega)\, G^*(\omega)\, d\omega \Rightarrow \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |g(\omega)|^2 d\omega$$

- heat equation $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ a - heat coefficient

↳ $u(x,t)$ is the temperature along a 1D rod

use spatial F.T. $u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(k,t) e^{-ikx} dk$

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} U(k,t) e^{-ikx} dk = a^2 \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} U(k,t) e^{-ikx} dk$$

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t} U(k,t) e^{-ikx} dk = \int_{-\infty}^{\infty} a^2 U(k,t) \frac{\partial^2}{\partial x^2}\left(e^{-ikx}\right) dk$$

- move partial derivatives into integral
- find $\frac{\partial^2 U}{\partial x^2}$
- divide by e^{-ikx} and remove integral

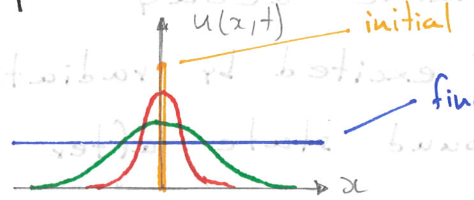
$$\frac{\partial U(k,t)}{\partial t} = -a^2 k^2 U(k,t)$$

solution $u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C(k) e^{-a^2k^2t} e^{-ikx} dx$ gaussian in k

↳ instead of solving DE we must compute F.T.

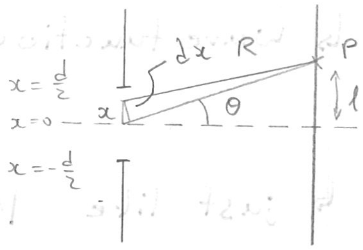
↳ apply heat at $x=0$

$C(x) = \delta(x)$ $C(k) = \text{constant}$



- F.T. can also be used to solve forced oscillator without having to guess the form of the solution

- diffraction $d\Psi(\theta) = a(x) \, e^{\left(i \frac{2\pi}{\lambda}(R + x\sin\theta)\right)} dx$



$d\Psi$ - amplitude contribution from aperture element dx

$a(x)$ - aperture dependent term

assume all rays attenuated the same: $d\Psi(\theta) = a(x) \, e^{\left(i \frac{2\pi}{\lambda} x \sin\theta\right)} dx$

use SAA $\sin\theta \approx \ell$: $d\Psi(\theta) = a(x) \, e^{\left(i \frac{2\pi}{\lambda} \ell x\right)} dx$

↳ define $k = \frac{2\pi}{\lambda} \ell$

$$\Psi(k) = A \int_{-\infty}^{\infty} a(x) e^{ikx} dx \implies \underline{\Psi(k) \sim \mathcal{F}(a(x))}$$

A - scaling factor

↳ gives intuative understanding of diffraction pattern

e.g., as slit width $\to 0$ diff pattern $\to$ constant

· Fraunhofer equation $\quad \Psi(l) \sim \mathcal{F}(a(x)) = \int_{-\infty}^{\infty} a(x)\, e^{ikx}\, dx$

$$k = \frac{2\pi}{\lambda} l$$

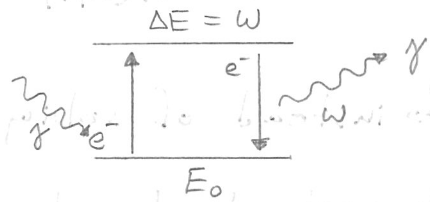
· intensity is given by $|\Psi(l)|^2$

↳ so observation of intensity gives energy of photons not their phase - incoherent measurement

↳ human eye sees the log of the intensity

- F.T. of a shifted delta function is a pure cosine

as shift ↑ the frequency ↑



- resonance decay

↳ e^- excited by radiation will return to ground state after time t by re-emitting photon

↳ probability distribution $\quad P(t) = \begin{cases} 0 & t<0 \\ e^{-\frac{t}{\tau}} & t>0 \end{cases} \qquad P(t) \sim |\Psi(t)|^2$

↳ wave function $\quad \Psi(t) = \begin{cases} 0 & t<0 \\ e^{-i\omega_0 t}\, e^{-\frac{t}{2\tau}} & t>0 \end{cases} \qquad \omega_0 = \frac{\Delta E}{\hbar}$

↳ just like $|\Psi(t)|^2$ gives probability distribution $|\phi(\omega)|^2$ in frequency domain gives energy distribution $I(\omega) \sim |\phi(\omega)|^2$

$$\phi(\omega) = \mathcal{F}(\Psi(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(t)\, e^{i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \frac{i}{\omega - \omega_0 + \frac{i\Gamma}{2}}$$

decay rate $\Gamma = \frac{1}{\tau}$

- Breit-Wigner distribution $\quad I(\omega) = |\phi(\omega)|^2 = \frac{1}{2\pi} \frac{1}{(\omega-\omega_0)^2 + \frac{\Gamma^2}{4}}$

↳ allows for a range of energies around a central maximum energy

- multidimensional Fourier transforms

↳ decompose using basis set for each dimension

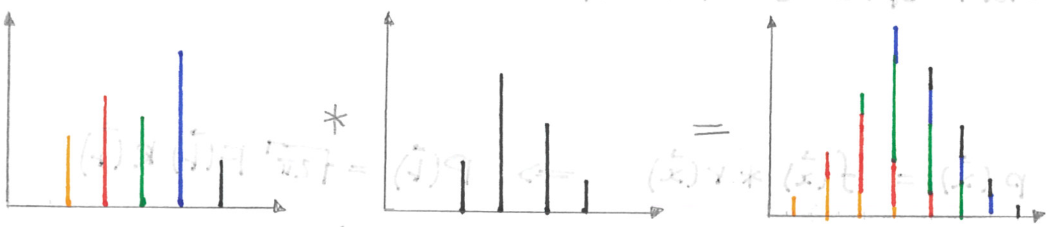
↳ introduce wave vector $\vec{k}$ where the phase gives the direction of the waves

$$g(k_x, k_y) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; f(x,y) \, e^{ik_x x} e^{ik_y y}$$

$$g(\vec{k}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\vec{r}) \, e^{i(\vec{k}\cdot\vec{r})} d^2r \implies \overbrace{g(\vec{k}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} f(\vec{r}) \, e^{i(\vec{k}\cdot\vec{r})} d^n r}^{\text{generally}}$$

- convolution

↳ applies distribution of one function at every point along another function and sums to give 3rd function



$$(f * g)(t) = \int_{-\infty}^{\infty} g(\tau) f(t-\tau) \, d\tau = \int_{-\infty}^{\infty} f(\tau) \, g(t-\tau) \, d\tau$$

$$(f * g)(t) = \mathcal{F}\left(\sqrt{2\pi} \, F(\omega) \, G(\omega)\right)(t)$$ — convolution is just the product in fourier domain

↳ convolution scales as N^2 (where N = no of discrete samples)

Fourier transform scales as N^2

FFT scales as $N \log(N)$ so used to speed up convolution

- properties of convolution

↳ commutative: $f * g = g * f$

↳ associative: $f * (g * h) = (f * g) * h = f * g * h$

- shifting functions using convolution

  $f(t) * \delta(t-d) = f(t-d)$ so convolving with offset δ gives offset f

- ideal measuring instrument samples as a delta function

  so output is convolution of measured signal with δ

- real instrument measures with some spread e.g. gaussian

  so output is convolution of measured signal with gaussian

  ↳ tends to smooth out oscillations

- point spread function

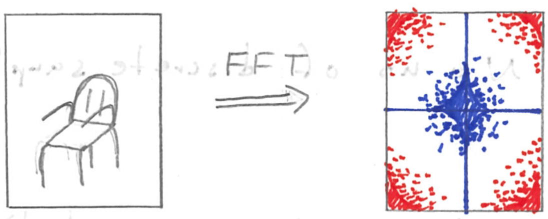
  ↳ for captured image the output is convolution of image with point spread function

- deconvolution

  captured image $p(\vec{x}) = f(\vec{x}) * r(\vec{x}) \Rightarrow P(\vec{k}) = \sqrt{2\pi}\, F(\vec{k})\, R(\vec{k})$

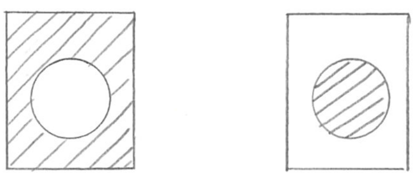
  estimate of true image $\tilde{f}(x) = \mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}} \frac{P(\vec{k})}{R(\vec{k})}\right)$

- jpg compression uses F.T.
- edge detection uses F.T.



● - low freq so long wavelengths gives rough shapes and shades

● - high freq so short wavelengths gives details and textures



low pass

high pass

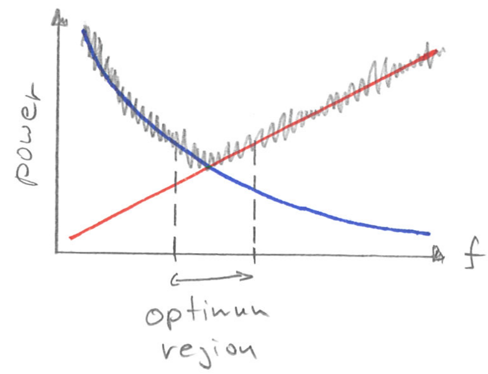
- pink noise / $\frac{1}{f}$ noise
  - ↳ $S(f) \propto \frac{1}{f^{\alpha}}$ $0 < \alpha < 2$
  - ↳ power spectral density (power per freq interval) falls off inversly proportionally to freq
  - ↳ caused by statistical fluctuations over long time (long term effects)
  - ↳ pink color in light spectrum

- white noise / f noise
  - ↳ constant power spectral density
  - ↳ random signal with equal intensity throught spectrum
  - ↳ white color in light spectrum

- Johnson - Niquist noise
  - ↳ white noise in electronics
  - ↳ caused by thermal agitation of charge carriers

$P = k_B T \Delta f$

P - noise powe
k_B - Boltzmann
Δf - bandwidth
T - temp in K



(blue) - $\frac{1}{f}$ noise due to long term drift
(red) - f noise due to thermal effects

design experiments so that measured frequency lies in the minimum region

- octave
  - ↳ interval between musical pitch and another one with double its frequency e.g.) 440Hz and octave above 880Hz
  - ↳ number of octaves between f_2 and f_1 $n = \log_2\left(\frac{f_2}{f_1}\right)$