Mechanics 2021

- suvat: $v = u + at$ $v^2 = u^2 + 2as$ $s = ut + \frac{1}{2}at^2$ $s = vt - \frac{1}{2}at^2$ $s = \frac{1}{2}(u+v)t$

- inertial reference frame
  ↳ frame of reference in which newtons 1st law holds
  ↳ not undergoing acceleration so momentum of COM constant

- newtons laws of motion
  ↳ 1.) body on which no force acts remains at rest or constant $\vec{v}$
  2.) rate of change of momentum is equal to force applied
  3.) when two bodies interact they excert equal and opposite forces on eachother

- conservation of linear momentum
  ↳ total momentum of an isolated system is always conserv

- two forces acting on same body are never action reactio pair

- impulse $\Delta \vec{p} = \int_{t_i}^{t_f} \vec{F}(t)\,dt$

- Work - energy theorem

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v = \frac{1}{2}\frac{d}{dx}(v^2)$$

$$\frac{1}{2}\frac{d}{dx}(v^2) = \frac{1}{2}(2v)\frac{dv}{dx} = v\frac{dv}{dx}$$

$$F = ma$$
$$= \frac{1}{2}m\frac{d}{dx}(v^2)$$
$$= \frac{d}{dx}\left(\frac{1}{2}mv^2\right)$$

$$\int F\,dx = \frac{1}{2}mv^2$$

$$W = \frac{1}{2}mv^2$$

- power $P = \vec{F}\cdot\vec{v}$

- potential energy function $U(x)$ $\quad F(x) = -\frac{dU}{dx}$

$$U(x) = -\int F(x)\,dx$$

- conservative force field
  ↳ any force field which depends only on x $\quad \vec{F} = \vec{F}(x)$
  ↳ any conservative force has potential energy $U(x)$
  ↳ motion in conservative force field conserves total energy

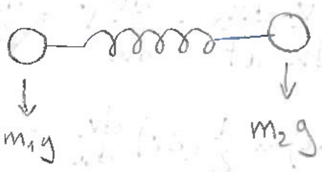
- viable region
  ↳ region in which particle can exist given its energy

- small oscillations about eql $\quad U(x) \approx U(x_0) + \frac{1}{2}\left|\frac{d^2U}{dx^2}\right|_{x_0}(x-x_0)^2$
  ↳ differentiating $\Rightarrow$ $F(x) = -\left|\frac{d^2U}{dx^2}\right|_{x_0}(x-x_0)$ — Hookes law

- SHM
  let $s = \left|\frac{d^2U}{dx^2}\right|_{x_0}$ $\quad m\frac{d^2x}{dt^2} = -sx$ $\quad \frac{d^2x}{dt^2} = -\omega^2 x$ $\quad \omega = \sqrt{\frac{s}{m}}$

- centre of mass: $\vec{R} = \frac{\Sigma m\vec{r}}{\Sigma m}$ $\quad M\ddot{\vec{R}} = \vec{F}^{ext}$

- reduced mass:

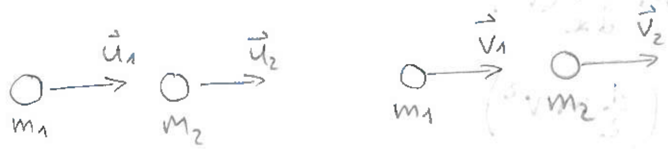


$$\ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 = F_{12}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

$$\ddot{\vec{r}}\mu = \vec{F}_{12} \qquad \mu = \frac{m_1 m_2}{M}$$

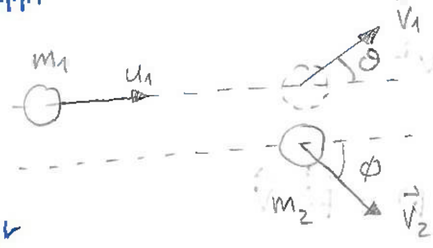
  ↳ external force cancels out only if proportional to mass
  ↳ only works for 2 particles

- inellastic collision in 1D



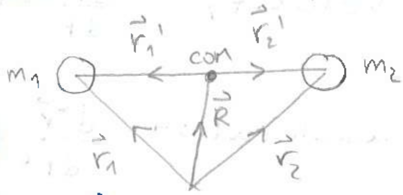
$$v_2 - v_1 = -e(u_2 - u_1) \qquad e = \sqrt{\frac{kE_f}{kE_i}} = \sqrt{\frac{|v_{sep}|}{|v_{app}|}}$$

- scattering from stationary target
  Θ - scattering angle
  ϕ - recoil angle $\quad$ b - impact parameter



- ellastic collision of equal masses. $\theta + \phi = 90°$

- COM frame

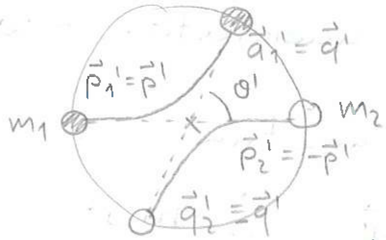


↳ in absence of external force COM is inertial

$\vec{r}_1{}' = \vec{r}_1 - \vec{R}$ $\quad \vec{r}_2{}' = \vec{r}_2 - \vec{R}$ $\quad \vec{p}_1{}' = \frac{m_1 m_2}{M}(\dot{\vec{r}}_1 - \dot{\vec{r}}_2) = -\mu\dot{\vec{r}}$ $\quad \vec{p}_2{}' = \mu\dot{\vec{r}}$

- KE in COM frame

$K = K' + \frac{1}{2}M\dot{\vec{R}}^2 \quad K' = \frac{1}{2}\mu\dot{\vec{r}}^2$



- scattering in COM frame

↳ by symmetry $\vec{p}_1{}' = \vec{p}\,'$ $\quad \vec{p}_2{}' = -\vec{p}\,'$ $\quad \vec{q}_1{}' = \vec{q}\,'$ $\quad \vec{q}_2{}' = -\vec{q}\,'$

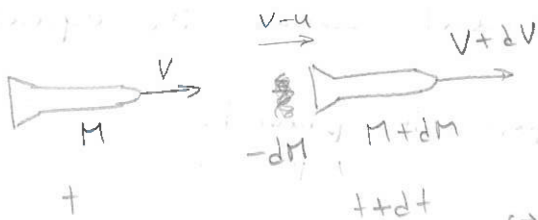
↳ fully characterised by $\vec{p}\,'$ and θ'

$\vec{p}_1 = \vec{p}\,'\left(\frac{m_1}{m_2} + 1\right) \quad \vec{p}_2 = 0 \quad \dot{\vec{R}} = \frac{\vec{p}\,'}{m_2} \qquad \phi = \frac{1}{2}(\pi - \theta')$

$\vec{q}_1 = \vec{q}\,' + \frac{m_1}{m_2}\vec{p}\,' \quad \vec{q}_2 = -\vec{q}\,' + \vec{p}\,' \qquad \tan\theta = \frac{\sin\theta'}{\frac{m_1}{m_2} + \cos\theta'}$

↳ K transfer $\left(\frac{K_2^f}{K_1}\right)_{max} = \frac{4m_1m_2}{(m_1+m_2)^2}$ in head on collision

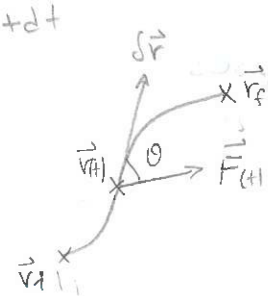
$\left(\frac{K_2^f}{K_1}\right)_{max} = \frac{4m_1m_2}{(m_1+m_2)^2}\sin^2\left(\frac{\theta'}{2}\right)$



- rocket equation

$M\frac{dv}{dt} = -u\frac{dM}{dt} \Rightarrow V(t) = V_0 + u\ln\left(\frac{M_0}{M(t)}\right)$

- Work energy theorem in 3D $\quad K_f - K_i = \int_{\text{path } \vec{r}(t)} \vec{F}\cdot d\vec{r}$

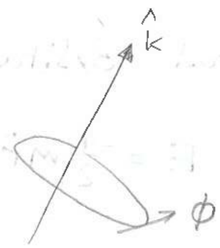


↳ cant apply potential function

only if $\text{curl}(\vec{F}(\vec{r}\,')) = 0$

- time derivative of time dependant vector

$$\frac{d\vec{a}}{dt} = \underbrace{\dot{a}\hat{a}}_{\text{along } \hat{a}} + \underbrace{a\vec{\omega}\times\hat{a}}_{\text{perpendicular to } \hat{a}} \qquad \vec{\omega} = \omega\hat{\omega} = |\dot{\phi}|\hat{k}$$



- velocity $\frac{d\vec{r}}{dt} = \vec{v} = \dot{r}\hat{r} + r\omega\hat{\phi}$ radial $= \dot{r}$ angular $= \omega r$

- acceleration $\frac{d^2\vec{r}}{dt^2} = \vec{a} = (\ddot{r} - \omega^2 r)\hat{r} + (2\omega\dot{r} + \dot{\omega}r)\hat{\phi}$

  radial $= \ddot{r} - \omega^2 r$ angular $= 2\omega\dot{r} + \dot{\omega}r$

- centripetal acceleration $= \omega^2 r = \frac{V_\phi^2}{r}$

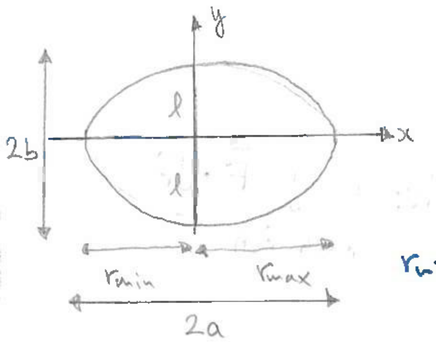
- angular momentum
  - ↳ $\vec{L} = m\vec{r}\times\vec{v}$ $\quad \vec{L} = I\omega$
  - ↳ for central force L is conserved $\frac{dL}{dt} = 0$ (no torque)

- Torque
  - ↳ $\vec{G} = \frac{d\vec{L}}{dt}$ $\quad \vec{G} = \vec{r}\times\vec{F}$

- motion in a central force field is allways in a plane

- Keplers laws
  - ↳ 1st planets move in ellipses with sun at one focus
  - 2nd radius vector from sun to planet sweeps out equal areas in equal times $\frac{T^2}{a^3} = \frac{4\pi^2}{GM} =$ constant
  - 3rd for all planets $\longrightarrow$ $a =$ radius of orbit

- ellipses



a - semi major axis
b - semi minor axis
ℓ - semi latus rectum

$r = \frac{\ell}{1 - e\cos\phi}$

r_{min} - periapsis $\quad r_{max}$ - apoapsis

- total orbital energy

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

- gravitational orbits
  - conserve energy, conserve angular momentum, are planar

- centrifugal potential : $\frac{L^2}{2mr^2} = \frac{1}{2} m V_\phi^2$
  - since L is constant $\frac{L^2}{2mr^2}$ is a function of r like a potential
  - as planet approaches star $r\downarrow$ so V_ϕ must increase (orbits faster
  - effective potential $U_{eff} = \frac{L^2}{2mr^2} - \frac{GMm}{r}$

centripetal force

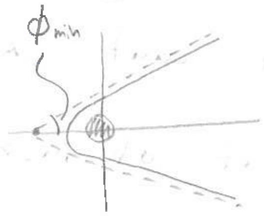
$$m\ddot{r} = -\frac{dU_{eff}}{dt} \Rightarrow m\ddot{r} = \frac{L^2}{mr^3} - \frac{GMm}{r} = \frac{(m\omega r^2)^2}{mr^3} - \frac{GMm}{r} = m\omega^2 r - \frac{GMm}{r}$$

- Circular orbit
  - lowest energy orbit $\left.\frac{dU_{eff}}{dr}\right|_{r_{eq}} = 0$ there is no radial force
    for a given L. $r_{eq} = \frac{L^2}{GMm^2}$

- elliptical orbit
  - bit more E than circular $U_{min} < E < 0$
  - moves between r_{min} and r_{max} at these points $\dot{r} = 0$ radial dir changes

$$a = \frac{1}{2}(r_{min} + r_{max}) = \frac{\ell}{(1+e)(1-e)} \qquad E = -\frac{GMm}{2a}$$

$$a = \frac{L^2}{GMm^2(1-e^2)} \qquad e = \sqrt{1 + \frac{2EL^2}{GMm^3M^2}} \, ?$$



$e > 1$

- hyperbolic orbits
  - $E > 0$ so planet is not in a closed orbit escapes to ∞
    $\phi_{min} = \cos^{-1}\left(\frac{1}{e}\right)$

- for a body rotating freely axis of rotation must pass through the COM (otherwise force must be exerted)

$$\vec{G}' = \frac{d\vec{L}'}{dt} \qquad \vec{G}' = \vec{r}' \times \vec{F}$$

- angular momentum in COM $\vec{L} = \vec{L}' + \vec{R} \times M\vec{R}$

- torque in COM $\vec{G} = \vec{G}' + \vec{R} \times \vec{F}$

- rigid bodies - move and rotate but relative positions of particles within them are fixed

- angular momentum of rigid body
  - ↳ $\vec{\omega}_i$ and $\vec{L}_i$ arent parallel only in special cases with symmetry where they cancel out so overall $\vec{\omega} \parallel \vec{L}$

- principal axis
  - ↳ for any wonky object there are 3 principal axis about which if it rotates then $\vec{L}$ is parallel to $\vec{\omega}$

  $\vec{L} = I\vec{\omega}$ I is a 3×3 moment of inertia tensor
  - ↳ turned into eigenvalue equation to find I scalars

- rotation about an axle $\frac{dL_{\parallel}}{dt} = G_{\parallel}^{ext}$
  - ↳ place the origin at the axle
  - ↳ axle cant excert force abouts its own axis

- moment of inertia $I = \sum m_i r_{i\perp}^2$ $I = \int_{body} r_{\perp}^2 \rho(\vec{r})\, dV$

- rotational kinetic energy $K = \frac{1}{2} I \omega^2$

- rotational work - energy theorem $\frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2 = \int_{\phi_i}^{\phi_f} G_{\parallel}(\phi)\, d\phi$

- parallel axis theorem
  - ↳ if I' is the moment of inertia about COM

  $I = I' + Md^2$

- angular frequency of precession $\frac{d\phi}{dt} = \frac{Mgr}{I\omega}$
  - ↳ independent of tilt
  - ↳ friction $\downarrow \omega$ so $\frac{d\phi}{dt} \uparrow$

- rotation of earth
  - ↳ rotating unit vectors $\vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} = x'(t)\hat{i}' + y'(t)\hat{j}' + z'(t)\hat{k}'$

    but $\vec{v}'$ and $\vec{a}'$ are different

  $\vec{v}' = \vec{v} - \Omega \times \vec{r}$ $\quad \vec{a} = \vec{a}' + 2\vec{\Omega} \times \vec{v}' + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$

- centrifugal force $F_{centrifugal} = -m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$

- coriolis force $F_{coriolis} = -2m\vec{\Omega} \times \vec{v}'$

  overall $m\vec{a}' = F \underbrace{- 2m\vec{\Omega} \times \vec{v}}_{\text{coriolis}} \underbrace{- m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})}_{\text{centrifugal}}$